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Mirror Symmetry and Noncommutative Geometry

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Outline

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Mirror symmetry for Calabi-Yau manifolds

Let X be a weak Calabi-Yau manifold, i.e. a compact complex manifold X with a trivial canonical bundle. It is believed that for any Kahler class ω on X and a class $B \in H^2(X, U(1))$ there is a $N = 2$ superconformal field theory which describes superstring propagation on X . Heuristically, this $N = 2$ SCFT is obtained by quantizing the supersymmetric sigma-model with target X . The metric is determined by the Kahler class, and the B-field is $B \in H^2(X, U(1))$.

We will call a triple (X, ω, B) a physicist's Calabi-Yau.

By definition, the chiral algebra of an $N = 2$ SCFT includes two copies of the $N = 2$ super-Virasoro algebra. The corresponding currents are

- $T(z)$ and $\bar{T}(\bar{z})$ - left- and right-moving stress-energy
- $Q_{\pm}(z)$ and $\bar{Q}_{\pm}(\bar{z})$ - left- and right-moving supercharges
- $J(z)$ and $\bar{J}(\bar{z})$ - left- and right-moving R-currents

Mirror involution is an automorphism of this current algebra defined by

$$\begin{aligned} T &\rightarrow T, & Q_{\pm} &\rightarrow Q_{\mp}, & J &\rightarrow -J, \\ \bar{T} &\rightarrow \bar{T}, & \bar{Q}_{\pm} &\rightarrow \bar{Q}_{\pm}, & \bar{J} &\rightarrow \bar{J}. \end{aligned}$$

It can happen that two distinct physicist's Calabi-Yaus lead to isomorphic $N = 1$ SCFTs. This means that there is an isomorphism of "Hilbert" spaces which preserves the OPE and acts on the generators of left/right $N = 1$ super-Virasoro algebra as the identity map.

In this case there are two possibilities:

- The isomorphism acts on the generators of the left/right $N = 2$ super-Virasoro as the identity map (i.e. it is an isomorphism of $N = 2$ SCFTs).
- This isomorphism acts on the generators of the left/right $N = 2$ super-Virasoro as the mirror involution.

In the first case, we will say that the two Calabi-Yau's are **equivalent**. In the second case they are called **mirror** to each other. The corresponding morphism of $N = 2$ SCFTs is called a mirror morphism.

Topological BPS D-branes

A D-brane is a boundary condition for the sigma-model which preserves $N = 1$ super-Virasoro. Topological BPS D-branes on a Calabi-Yau are D-branes which preserve $N = 2$ super-Virasoro.

There are two kinds of topological D-branes which are called A-branes and B-branes.

- A-branes: on the boundary $z = \bar{z}$ we have

$$T = \bar{T}, J = -\bar{J}, Q_{\pm} = \bar{Q}_{\mp}.$$

- B-branes: on the boundary

$$T = \bar{T}, J = \bar{J}, Q_{\pm} = \bar{Q}_{\pm}.$$

In fact, both A-branes and B-branes form additive categories. By definition, morphisms between a pair of branes (of the same kind) are BPS states of the string stretched between them. Topological string amplitudes on a disc compute compositions of morphisms.

Let us denote these categories \mathcal{A} and \mathcal{B} . It is widely believed that \mathcal{A} and \mathcal{B} depend only on the $N = 2$ SCFT, and not on the Calabi-Yau itself. Thus

- If (X, ω, B) and (X', ω', B') are equivalent, then $\mathcal{A} \sim \mathcal{A}'$ and $\mathcal{B} \sim \mathcal{B}'$.
- If (X, ω, B) and (X', ω', B') are mirror, then $\mathcal{A} \sim \mathcal{B}'$ and $\mathcal{B} \sim \mathcal{A}'$.

In addition, one can show that \mathcal{A} does not depend on the complex structure on X , while \mathcal{B} does not depend on ω and the $(1, 1)$ part of the B-field.

Homological Mirror Symmetry

What is the geometric significance of the statement that X and X' are equivalent (resp. mirror)?

Kontsevich proposed the following answers:

- If (X, ω, B) is equivalent to (X', ω', B') , then the bounded derived category $D^b(X)$ is equivalent to $D^b(X')$.
- If (X, ω, B) is mirror to (X', ω', B') , then $D^b(X)$ is equivalent to the so-called Fukaya category of (X', ω', B') , and vice versa.

Let us decipher these claims.

A coherent sheaf is a thing which locally looks like a cokernel of a map between finite-dimensional holomorphic vector bundles. Coherent sheaves form an additive (actually, an abelian) category denoted $Coh(X)$.

Let $Komp^b(X)$ be the category of bounded complexes of coherent sheaves. An object of $Komp^b(X)$ looks like

$$\dots \rightarrow A^{-1} \rightarrow A^0 \rightarrow A^1 \rightarrow A^2 \rightarrow \dots,$$

where the composition of any two consecutive maps is zero.

A morphism of complexes induces a morphism of their cohomologies. A morphism in $Komp^b(X)$ is called a quasi-isomorphism if it induces an isomorphism of cohomologies.

$D^b(X)$ is the category of complexes modulo complexes quasi-isomorphic to zero. $D^b(X)$ is much nicer than $Coh(X)$ and sometimes can even be explicitly described.

The Fukaya category $F^b(X)$ has not been completely constructed yet. Its objects are supposed to be special Lagrangian submanifolds equipped with flat unitary connections.

Morphisms have been completely described only for pairs of special Lagrangian submanifolds which intersect transversally. Roughly speaking, the space of morphisms is \mathbb{C}^n , where n is the number of intersection points. Composition of morphisms is computed by summing over holomorphic maps from a disc into X such that the boundaries of the disc map to special Lagrangian submanifolds. More on this later.

The most important point here is that $D^b(X)$ does not depend on ω and B , while $F^b(X)$ supposedly does not depend on the complex structure on X .

It is very tempting to make the following conjectures:

- For any (X, ω, B) $D^b(X)$ is equivalent to the category of B-branes \mathcal{B} .
- For any (X, ω, B) $F^b(X)$ is equivalent to the category of A-branes \mathcal{A} .

Then Kontsevich's conjecture follows from the string theory lore described above.

Some immediate puzzles:

- Both $D^b(X)$ and $F^b(X)$ are triangulated categories. Is there a natural triangulated structure on \mathcal{A} and \mathcal{B} ?
- Neither $D^b(X)$ nor $F^b(X)$ depend on the B-field, while both \mathcal{A} and \mathcal{B} have a nontrivial dependence.

It appears that one can make triangulated categories out of \mathcal{A} and \mathcal{B} using the information contained in n -point topological correlators on a disc.

The second puzzle is resolved by modifying the definition of $D^b(X)$ and $F^b(X)$ in a way which depends on the B-field. Let us explain how to do this for $D^b(X)$.

Given $B \in H^2(X, U(1))$ we can define a noncommutative algebraic variety which is in some sense a “deformation” of X . This is easiest to see when B is a torsion element. Then there exists a cover $\{U_\alpha\}, \alpha \in K$ and a 1-cochain $\{R_{\alpha\beta}(z)\}$ with values in $GL(n, \mathbb{C})$ such that the 2-cocycle

$$\zeta_{\alpha\beta\gamma} = R_{\alpha\beta}(z)R_{\beta\gamma}(z)R_{\gamma\alpha}(z)$$

takes values in $U(1)$ and is cohomologous to $B \in H^2(X, U(1))$.

Using the 1-cochain $\{R_{\alpha\beta}(z)\}$ as gluing functions on double overlaps, we can define a sheaf of noncommutative algebras on X which is locally isomorphic to the algebra of $GL(n, \mathbb{C})$ -valued holomorphic functions. Such an algebra is called an Azumaya algebra over X and will be denoted \mathfrak{A}_B .

There is some arbitrariness in the above construction of \mathfrak{A}_B , but once B is fixed, different \mathfrak{A}_B are Morita equivalent. This means that the category of modules over \mathfrak{A}_B is independent of any choices made in the construction of \mathfrak{A}_B .

The algebra \mathfrak{A}_B (or rather the category of modules over it) should be regarded as a noncommutative “deformation” of X . There is also a corresponding derived category $D^b(\mathfrak{A}_B)$.

More generally, when B is not a torsion class, one can canonically associate to it a gerbe on X . The derived category of coherent sheaves on this gerbe is a natural generalization of $D^b(X)$ and $D^b(\mathfrak{A}_B)$.

We get a refined version of the first Kontsevich's conjecture:

- If (X, ω, B) is equivalent to (X', ω', B') , then $D^b(\mathfrak{A}_B)$ is equivalent to $D^b(\mathfrak{A}_{B'})$.

We can check this for complex tori, since both the $N = 2$ SCFT and the derived category are well understood in this case. A check is also possible for K3, where the derived category is understood for $B = 0$, and there is some control over the SCFT as well.

N=1 SCFTs associated to flat tori

Let (T, G, B) be a real torus equipped with a flat metric G and a B-field. There is a well-known criterion for two such tori to give equivalent $N = 1$ SCFTs. Let

$T = \mathbb{R}^n / \Gamma, T^* = \mathbb{R}^n / \Gamma^*$, and let us define a metric on $T \times T^*$ as follows:

$$\mathcal{G} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}.$$

Let q be a natural even symmetric bilinear form on $\Gamma \oplus \Gamma^*$ given by

$$q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Two tori T and T' give the same $N = 1$ SCFT (i.e. are T-dual) iff there is an isomorphism of lattices $\Gamma \oplus \Gamma^*$ and $\Gamma' \oplus \Gamma'^*$ which intertwines q and q', \mathcal{G} and \mathcal{G}' .

N=2 SCFTs associated to complex tori

Given (T, G, B) and a complex structure I on T compatible with G one can define an $N = 2$ SCFT. The corresponding R-currents are simply

$$J(z) = \omega_{ij} \psi^i(z) \psi^j(z), \bar{J}(\bar{z}) = \omega_{ij} \bar{\psi}^i(\bar{z}) \bar{\psi}^j(\bar{z}),$$

where $\omega = GI$ is the Kähler form.

We have found a criterion for (T, I, ω, B) and (T', I', ω', B') to be equivalent or mirror-symmetric. Let us define a pair of commuting complex structures on $T \times T^*$:

$$\mathcal{I}(I, B) = \begin{pmatrix} I & 0 \\ BI + I^t B & -I^t \end{pmatrix},$$

$$\mathcal{J}(\omega, B) = \begin{pmatrix} \omega^{-1} B & -\omega^{-1} \\ \omega + B\omega^{-1} B & -B\omega^{-1} \end{pmatrix}.$$

Precisely when B is of type $(1, 1)$, \mathcal{I} reduces to the obvious product complex structure on $T \times T^*$.

The criteria:

- (T, I, ω, B) and (T', I', ω', B') are equivalent iff there exists an isomorphism of lattices $\Gamma \oplus \Gamma^*$ and $\Gamma' \oplus \Gamma'^*$ which intertwines q and q' , \mathcal{I} and \mathcal{I}' , \mathcal{J} and \mathcal{J}' .

- (T, I, ω, B) and (T', I', ω', B') are mirror iff there exists an isomorphism of lattices $\Gamma \oplus \Gamma^*$ and $\Gamma' \oplus \Gamma'^*$ which intertwines q and q' , \mathcal{I} and \mathcal{J}' , \mathcal{J} and \mathcal{I}' .

On the other hand, we have a theorem of Polischuk:

- If both T and T' are algebraic, both B and B' are torsion, and there exists an isomorphism of lattices $\Gamma \oplus \Gamma^*$ and $\Gamma' \oplus \Gamma'^*$ which intertwines q and q' , \mathcal{I} and \mathcal{I}' , then $D^b(\mathfrak{A}_B)$ is equivalent to $D^b(\mathfrak{A}'_{B'})$.

Hence for algebraic tori with torsion B-field isomorphism of $N = 2$ SCFTs implies equivalence of the derived categories of Azumaya algebras. This proves the first part of the generalized Kontsevich's conjecture for algebraic tori.

Remarks:

- I believe that Polischuk's theorem holds also for non-algebraic tori and non-torsion B-fields.
- Our result means that in general B-branes are related to coherent sheaves on a noncommutative “deformation” of X . $Coh(X)$ is relevant iff B is of type $(1, 1)$.
- There is a theorem by Orlov which describes when two K3 surfaces have equivalent derived categories. Using his result and some results of Aspinwall and Morrison on the K3 SCFT, one can check that the first part of Kontsevich's conjecture holds for algebraic K3s and B-field of type $(1, 1)$.

In the mathematical literature there are some incomplete results on $D^b(\mathfrak{A}_B)$ for K3. The criterion of equivalence of $N = 2$ SCFTs for K3s when B not of type $(1, 1)$ is not completely clear to me either.

We also obtain a converse result:

- If two Azumaya algebras over algebraic tori have equivalent derived categories, then for any flat Kähler metric on the first torus there exists a flat Kähler metric on the second torus such that the corresponding $N = 2$ SCFTs are isomorphic.

Even though we do not yet know the complete definition of the Fukaya category, our results imply:

- Two symplectic tori equipped with a B-field have equivalent Fukaya categories iff there is an isomorphism of lattices $\Gamma \oplus \Gamma^*$ and $\Gamma' \oplus \Gamma'^*$ which intertwines q and q' , \mathcal{J} and \mathcal{J}' .

Fukaya category in more depth

For $B = 0$ an object of the Fukaya category consists of

- A special Lagrangian submanifold (SLAG) L .
- A flat vector bundle on L such that the eigenvalues of the monodromy have unit modulus.
- A certain integer-valued datum.

In physical terms, the integer-valued datum is the choice of the branch of $\arg Z$, where Z is the BPS central charge of the brane.

In mathematical terms, one needs to consider the restriction of the holomorphic section of the canonical bundle to L . This is a complex constant Z times the volume form, and one needs to choose the branch of $\arg Z$.

The space of morphisms between two objects is a complex of vector spaces, i.e. a graded vector space endowed with a morphism d , $d^2 = 0$. If the objects intersect transversally, then the space of morphisms is

$$\bigoplus_i V_i,$$

where the sum is over the intersection points, $V_i \simeq \mathbb{C}$, and $\deg V_i$ is the Maslov index of the intersection.

I will not explain how to define d on this graded vector space. or how to define the composition of morphisms. I only note that in this way one obtains an A_∞ category rather than a true category. To get a true category, one may, for example, take the cohomology of the above complexes.

Fukaya category with a B-field

If $B \neq 0$, the definition of an object should be modified. Instead of a flat vector bundle on L one needs to consider a projectively flat vector bundle whose connection A satisfies

$$F_A = 2\pi B|_L \cdot id,$$

where B is the lift of the B-field from $H^2(X, U(1))$ to $H^2(X, \mathbb{R})$.

This follows from the gauge-invariance

$$B \rightarrow B + d\lambda, \quad A \rightarrow A + 2\pi\lambda|_L \cdot id.$$

Since periods of F_A are quantized, and periods of B are not, not every SLAG L admits such a vector bundle. Hence switching on the B-field reduces the number of objects in $F^b(X)$.

Fukaya category and noncommutative geometry

It seems that Fukaya category should admit a more algebraic definition. Indeed, it is believed that the Fukaya category is equivalent to the category of A-branes. The latter should be definable given an algebraic datum, namely an $N = 2$ SCFT.

If such an algebraic description exists, it should be noncommutative even for $B = 0$. I.e. A-branes must be modules over some noncommutative algebra.

Here is some evidence.

- A flat connection on L is a holonomic D-module on L . I remind that a D-module on L is a module over the algebra of differential operators on L .

- A projectively flat connection with $F_A = 2\pi B|_L$ is a holonomic module over the algebra of twisted differential operators. “Twisted” means that we set $[\partial_j, \partial_k] = 2\pi i B_{jk}$ instead of $[\partial_j, \partial_k] = 0$.
- A D-module M on a manifold Y has a characteristic variety $Char(M) \subset T^*Y$. For a holonomic D-module this variety is Lagrangian. This suggests that we regard the Calabi-Yau X as a “compactification” of T^*Y for some Lagrangian $Y \subset X$. There should also be a projection $X \rightarrow Y$ whose fibers are Lagrangian tori (like in Strominger-Yau-Zaslow).
- This suggests that L is $Char(M)$, where M is a module over the quantization of X . The fact that Fukaya category for tori resembles the category of modules over a noncommutative torus has been also noticed by Soibelman.

- At special points on the Kähler moduli space there exist A-branes which ARE NOT special Lagrangian. This was first noticed by Ooguri, Oz and Yin. If $\dim_{\mathbb{C}} X = d$, an A-brane may have real dimension $d + 2k$, where k is a non-negative integer.

For example, if d is even, one may have an A-brane which wraps the whole X . For this to work, the curvature of the line bundle on X must satisfy

$$(F_A \cdot \omega^{-1})^2 = -id.$$

Since periods of F_A are quantized, this condition can be satisfied only for very special ω .

It would be very interesting to understand the significance of this condition in algebraic terms.

Conclusions

- In the presence of a B-field not of type $(1, 1)$, B-branes should be thought of as objects of the derived category of a noncommutative “deformation” of the Calabi-Yau (an Azumaya algebra, or more generally a gerbe over a Calabi-Yau).
- The definition of the Fukaya category is modified in the presence of a B-field (flat connections are replaced by projectively flat connections).
- A-branes on a Calabi-Yau X seem to be closely related to modules over the quantization of X .