

Elements of Topological M-Theory

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Preface

The **topological string** on a Calabi-Yau threefold X is (loosely speaking) an “integrable spine” of the Type II string theory on $X \times \mathbb{R}^{3,1}$. Calabi-Yau spaces are the natural target space because they preserve some supersymmetry.

The full Type II theory in 10 dimensions is known to develop an 11-dimensional Poincare invariance at strong coupling — leading to the conjecture that there is an **M-theory** in 11 dimensions whose low energy limit is 11-dimensional supergravity.

Could something similar happen for the topological string — could the Calabi-Yau space X grow an extra dimension? The natural target spaces in this case would be G_2 holonomy manifolds. But what is the appropriate low energy action?

Outline

Stable forms and Hitchin's functionals

Relation to the topological string

Topological M-theory?

Geometric structures and forms

Hitchin introduced new **action functionals** for which the critical points are geometric structures on a manifold X — e.g. symplectic structure, complex structure, G_2 holonomy metric.

The construction is based on the idea that geometric structures are often characterized by the existence of particular **differential forms** on X — e.g. presymplectic form ω , holomorphic volume form Ω , associative 3-form Φ — obeying some **integrability conditions** — e.g. $d\omega = 0$, $d\Omega = 0$, $d\Phi = d * \Phi = 0$.

Stable forms

A form ω on an n -dimensional manifold X can give rise to a geometric structure because it defines a **reduction** of the group $GL(n, \mathbb{R})$ of coordinate changes (structure group of TX) to the subgroup that preserves ω .

In order to get the same structure at every point of X , independent of small perturbations of ω , want ω to be **nondegenerate** and **generic** in an appropriate sense. What does this mean for a general p -form?

Stable 2-forms

If $p = 2$, we know how to define a nondegenerate form: it is $\omega = M^{ij} dx_i \wedge dx_j$ with $\det M \neq 0$.

A nondegenerate real 2-form in dimension $n = 2m$ can always locally be written

$$\omega = e_1 \wedge f_1 + \cdots + e_m \wedge f_m,$$

for some choice of basis $\{e_1, \dots, e_m, f_1, \dots, f_m\}$ for T^*X , varying over X (“vielbein”). So ω defines a **presymplectic** structure: reduces $GL(2m, \mathbb{R}) \rightarrow Sp(2m, \mathbb{R})$ at each point.

If $d\omega = 0$, then there exist local coordinates $(p_1, \dots, p_m, q_1, \dots, q_m)$ such that

$$\omega = dp_1 \wedge dq_1 + \cdots + dp_m \wedge dq_m.$$

Then ω defines a **symplectic** structure.

Stable forms

Another way of expressing the statement that a 2-form ω is nondegenerate is to say that any small perturbation $\omega \rightarrow \omega + \delta\omega$ can be undone by a local $GL(n, \mathbb{R})$ transformation. In this sense ω is **stable**.

This formulation can be generalized to other p -forms: we say $\omega \in \Omega^p(X, \mathbb{R})$ is **stable** if it lies in an open orbit of the local $GL(n, \mathbb{R})$ action, i.e. if any small perturbation can be undone by a local $GL(n, \mathbb{R})$ action.

So e.g. there are no stable 0-forms; any 1-form that is everywhere nonvanishing is stable; for 2-forms stability is equivalent to nondegeneracy (when n is even!)

Stable 3-forms

What about $p = 3$? The dimension of $\wedge^3(\mathbb{R}^n)$ grows like n^3 , but the dimension of $GL(n, \mathbb{R})$ grows like $n^2 \Rightarrow$ for large enough n , there **cannot** be stable 3-forms!

In large enough dimensions, **every 3-form is different**.

Stable 3-forms in dimension 6

But some **exceptional** examples exist — e.g. $n = 6$.

$$\dim \wedge^3(\mathbb{R}^6) = 20$$

$$\dim GL(6, \mathbb{R}) = 36$$

So consider a stable real 3-form ρ in dimension 6. The **stabilizer** of ρ inside $GL(6, \mathbb{R})$ has real dimension $36 - 20 = 16$; in fact it is either $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ or $SL(3, \mathbb{C}) \cup SL(3, \mathbb{C})$.

We're interested in the case of $SL(3, \mathbb{C}) \cup SL(3, \mathbb{C})$.

Stable 3-forms in dimension 6

If ρ has stabilizer $SL(3, \mathbb{C}) \cup SL(3, \mathbb{C})$ it can be written locally in the form

$$\rho = \frac{1}{2} (\zeta_1 \wedge \zeta_2 \wedge \zeta_3 + \bar{\zeta}_1 \wedge \bar{\zeta}_2 \wedge \bar{\zeta}_3)$$

where $\zeta_1 = e_1 + ie_2$, $\zeta_2 = e_3 + ie_4$, $\zeta_3 = e_5 + ie_6$, and the e_i are a basis for T^*X , varying over X . The ζ_i determine an **almost complex structure** on X . If we are lucky, there exist local complex coordinates (z_1, z_2, z_3) such that $\zeta_i = dz_i$; in that case we say the almost complex structure is **integrable**, i.e. it is an honest complex structure.

Stable 3-forms in $d = 6$

Given the stable $\rho \in \Omega^3(X, \mathbb{R})$

$$\rho = \frac{1}{2} (\zeta_1 \wedge \zeta_2 \wedge \zeta_3 + \bar{\zeta}_1 \wedge \bar{\zeta}_2 \wedge \bar{\zeta}_3)$$

we can define another real 3-form

$$\hat{\rho}(\rho) = \frac{i}{2} (\zeta_1 \wedge \zeta_2 \wedge \zeta_3 - \bar{\zeta}_1 \wedge \bar{\zeta}_2 \wedge \bar{\zeta}_3)$$

This $\hat{\rho}$ is **algebraically determined** by ρ . Then $\Omega = \rho + i\hat{\rho}$ is a **holomorphic 3-form** in the almost complex structure determined by ρ .

The complex structure is **integrable** just if $d\Omega = 0$, i.e. $d\rho = 0$, $d\hat{\rho}(\rho) = 0$.

Hitchin's holomorphic volume functional

The integrability condition $d\rho = 0$, $d\hat{\rho}(\rho) = 0$ can be obtained by **extremization** of a volume functional:

$$V_H(\rho) = \frac{1}{2} \int_X \hat{\rho}(\rho) \wedge \rho = \frac{-i}{4} \int \Omega \wedge \bar{\Omega}.$$

Here ρ varies **within a cohomology class**, $[\rho] \in H^3(X, \mathbb{R})$ — i.e. $\rho = \rho_0 + d\beta$ for some fixed closed ρ_0 . So $d\rho = 0$ of course; and the effect of variation of β is

$$\delta V_H(\rho) = \int_X \hat{\rho}(\rho) \wedge d(\delta\beta),$$

so $\delta V_H(\rho) = 0$ for all $\delta\beta \Rightarrow d\hat{\rho}(\rho) = 0$.

Intermezzo

One can write $V_H(\rho)$ more explicitly in terms of ρ :

$$V_H(\rho) = \frac{1}{2} \int d^6x \sqrt{\rho_{a_1 a_2 a_3} \rho_{a_4 a_5 a_6} \rho_{a_7 a_8 a_9} \rho_{a_{10} a_{11} a_{12}} \epsilon^{a_2 a_3 a_4 a_5 a_6 a_7} \epsilon^{a_8 a_9 a_{10} a_{11} a_{12} a_1}}$$

Hitchin's holomorphic volume functional

So extremization of V_H , with a fixed $[\rho] \in H^3(X, \mathbb{R})$, leads to **integrable complex structures** on X , equipped with holomorphic 3-forms Ω , such that the real parts of the periods are fixed by $[\operatorname{Re} \Omega] = [\rho]$. (Almost “Calabi-Yau structures” on X , except that we didn't say X was Kähler. No Ricci flat metrics here!)

Complex geometry emerges from real 3-forms! A peculiarity of $d = 6$.

Hitchin vs. the B model

So altogether we have

$$V_H(\rho) = \frac{-i}{4} \int \Omega \wedge \bar{\Omega},$$

the action of a “2-form abelian gauge theory” in 6 dimensions, for which the classical solutions are roughly Calabi-Yau structures. (A stripped-down **gravity theory**.)

We already know a theory in 6 dimensions with these classical solutions — the **B model topological string**, or “Kodaira-Spencer gravity.” So could V_H be a target space action for the B model? Define formally the partition function,

$$Z_H([\rho]) = \int_{\rho \in [\rho]} D\rho \exp(V_H(\rho)).$$

This is a real function of $[\rho] \in H^3(X, \mathbb{R})$. We want to compare it to the B model partition function.

The B model and background dependence

The B model partition function is naively a holomorphic function of the complex moduli of X , $Z_B(t)$. But more precisely, it has a **background dependence**: depends on choice of base-point $\Omega_0 \in H^3(X, \mathbb{C})$, so it should be written $Z_B^{\Omega_0}(t)$. Here t parameterizes tangent vectors to the extended Teichmuller space (complex structures together with a choice of holomorphic 3-form): $t \in H^{3,0}(X_{\Omega_0}, \mathbb{C}) \oplus H^{2,1}(X_{\Omega_0}, \mathbb{C})$. [Bershadsky-Cecotti-Ooguri-Vafa]

The various $Z_B^{\Omega_0}$ are related by a **holomorphic anomaly equation** which gives the parallel transport from one Ω_0 to another Ω'_0 . This equation has an elegant interpretation. [Witten]

Background dependence as the wavefunction property

Consider $H^3(X, \mathbb{R})$ as a symplectic vector space; then we can **quantize** it. (Think of \mathbb{R}^2 with $\omega = dp \wedge dq$.) The Hilbert space consists of functions ψ which depend on “half the coordinates,” e.g. functions on a Lagrangian subspace (choice of **polarization**). Different polarizations are related by Fourier-like transforms.

So we can have **real polarizations** (like $\psi(q)$ or $\psi(p) = \int dq e^{ipq} \psi(q)$), obtained by splitting $H_3(X, \mathbb{Z})$ into “A and B cycles” — symplectic marking,
$$H_3(X, \mathbb{Z}) = H_3(X, \mathbb{Z})_A \oplus H_3(X, \mathbb{Z})_B.$$

Can also have **holomorphic polarizations** (like $\psi(q + ip)$ or $\psi(q + \tau p)$) obtained by splitting $H^3(X, \mathbb{C})$, e.g. Hodge splitting
$$H^3(X, \mathbb{C}) = (H^{3,0} \oplus H^{2,1}) \oplus (H^{0,3} \oplus H^{1,2}).$$

The various $Z_B^{\Omega_0}$ are expressions of the **same wavefunction** in **different polarizations** of $H^3(X, \mathbb{R})$ — polarization given by the Hodge splitting determined by Ω_0 .

Hitchin vs. the B model

So we've seen that Z_B depends on only **half** the coordinates of $H^3(X, \mathbb{R})$ (Lagrangian subspace) and requires a **choice** of polarization. These features are visible already classically (genus zero).

On the other hand, Z_H is a function on **all** of $H^3(X, \mathbb{R})$, and doesn't seem to require any choice.

So we can't say $Z_H = Z_B$. Instead, propose that $Z_H = Z_B \otimes \overline{Z_B}$, or more precisely, Z_H is the **Wigner function** associated to Z_B : this is the phase-space density,

$$(Z_B \otimes \overline{Z_B})([\rho]) = \int d\Phi e^{-\langle \Phi, \rho_B \rangle} |Z_B(\rho_A + i\Phi)|^2.$$

Here we wrote Z_B in the real polarization determined by a choice of A and B cycles (symplectic marking): $\rho_A = \rho|_{H_3(X, \mathbb{R})_A}$,
 $\rho_B = \rho|_{H_3(X, \mathbb{R})_B}$.

Hitchin vs. the B model

The relation $Z_H = Z_B \otimes \overline{Z_B}$ holds at least classically — i.e. **leading order** asymptotics in large $[\rho]$ expansion. In that limit the **saddle-point** evaluation of

$$\int d\Phi e^{-\langle \Phi, \rho_B \rangle} |Z_B(\rho_A + i\Phi)|^2$$

indeed gives

$$(Z_B \otimes \overline{Z_B})([\rho]) \sim e^{i \int \Omega \wedge \overline{\Omega}},$$

where Ω is the complex structure with $[\operatorname{Re} \Omega] = [\rho]$.

This agrees with the saddle-point evaluation of $Z_H([\rho])$, almost by definition.

Hitchin vs. the B model

At **one-loop** the situation is subtler: careful BV quantization shows that, in order to agree with the known one-loop $Z_B \otimes \overline{Z_B}$, one needs to replace Z_H by an **extended** Z_H . [Pestun-Witten]

The extended Z_H includes fields describing variations of **generalized** complex structures. [Hitchin]

Hitchin B model and black holes

The Wigner function of the B model (which we have now identified as $Z_H([\rho])$) has appeared recently in another context: it was conjectured that $Z_H([\rho])$ computes the **number of states** of a **black hole**.

Motivation from the **attractor mechanism**: suppose we consider Type IIB superstring on $X \times \mathbb{R}^{3,1}$. Then we can construct a charged **black hole** by wrapping a D3-brane on a 3-cycle $Q \in H_3(X, \mathbb{Z})$. The complex moduli of X near the horizon then get fixed to an Ω satisfying $[\text{Re } \Omega] = Q^*$.

This is exactly the Ω that Hitchin's gauge theory constructs if we fix the class $[\rho] = Q^*$. And $Z_H([\rho])$ is exactly the number of states of the black hole!

[Ooguri-Strominger-Vafa]

So Z_H **reformulates** the B model in a way naturally adapted to the counting of black hole states.

Hitchin's symplectic volume functional

What about the A model? This has to do with variations of **symplectic structures**.

Hitchin introduced a functional which produces at its critical points symplectic structures in $d = 6$.

A **stable 4-form** σ in $d = 6$ may be written $\sigma = \frac{1}{2}k \wedge k$. Then k gives a **presymplectic structure**. Define

$$V_S(\sigma) = \frac{1}{6} \int k \wedge k \wedge k = \frac{1}{3} \int k \wedge \sigma.$$

Varying $V_S(\sigma)$ with $[\sigma] \in H^4(X, \mathbb{R})$ fixed, i.e. $\sigma = \sigma_0 + d\gamma$ for some closed σ_0 , get

$$\delta V_S = \frac{1}{2} \int k \wedge d\delta\gamma,$$

so $\delta V_S = 0 \Rightarrow dk = 0$, i.e. k defines a **symplectic structure**.

So V_S has the same classical solutions as the A model.

Hitchin A model and black holes

Another **attractor mechanism**: consider M-theory on $X \times \mathbb{R}^{4,1}$. Then we can construct a charged **black hole** by wrapping an M2-brane on a 2-cycle $Q \in H_2(X, \mathbb{Z})$. The Kähler moduli of X near the horizon then get fixed to an k satisfying $\frac{1}{2}[k \wedge k] = Q^*$.

This is exactly the k that Hitchin's gauge theory constructs if we fix the class $[\sigma] = Q^*$. So the weird fact that V_S involves the 4-form $k \wedge k$ instead of the 2-form k gets naturally related to the fact that we want to fix a 2-cycle charge.

The leading number of states of this black hole is $\sim \exp \int k \wedge k \wedge k$ at large k , which agrees with the classical value of the Hitchin A model partition function,

$$Z_S([\sigma]) = \int_{\sigma \in [\sigma]} D\sigma \exp(V_S(\sigma)).$$

Hitchin A model and black holes

It was known before that one can extract counts of BPS states of wrapped M2-branes in five dimensions from the perturbative A model.

[Gopakumar-Vafa]

Here we are finding that the Hitchin version of the A model is organized differently — its partition function seems to be **directly** counting the number of states.

Hitchin and topological strings

So Hitchin's functionals V_H and V_S in 6 dimensions seem to give **reformulations** of the target space dynamics of the B model and A model topological string theories, naturally adapted to the problem of counting black holes.

We only argued for this classically; it remains to be seen how much V_H and V_S can capture about the quantum theories.

These reformulations may be of interest in their own right. They are also naturally related to Hitchin's functional in 7 dimensions.

Stable 3-forms in dimension 7

Another **exceptional** example — $n = 7$.

$$\dim \wedge^3(\mathbb{R}^7) = 35$$

$$\dim GL(7, \mathbb{R}) = 49$$

So consider a stable real 3-form Φ in dimension 7. The **stabilizer** of Φ inside $GL(7, \mathbb{R})$ has dimension $49 - 35 = 14$; in one open subset it is the compact form of G_2 .

Stable 3-forms in dimension 7 and G_2 structures

If $\Phi \in \Omega^3(Y, \mathbb{R})$ is stable of the appropriate sort, it determines a “ G_2 structure” on Y (reduction of the structure group to G_2).

Concretely, Φ can be written in the form

$$\Phi = \sum_{i,j,k=1}^7 \Psi_{ijk} e_i \wedge e_j \wedge e_k,$$

where Ψ_{ijk} are the structure constants of the imaginary octonions, and the e_i are a basis for T^*Y , varying over Y . G_2 occurs as the automorphism group of the imaginary octonions.

We can construct a **metric** from Φ , namely

$$g_\Phi = \sum_{i=1}^7 e_i \otimes e_i.$$

This metric has G_2 holonomy just if $d\Phi = 0$, $d *_\Phi \Phi = 0$.

Hitchin's G_2 volume functional

The integrability condition $d\Phi = 0$, $d *_{\Phi} \Phi = 0$ can – again – be obtained by **extremization** of the volume functional:

$$V_7(\Phi) = \int_Y \Phi \wedge *_{\Phi} \Phi.$$

Again Φ varies **within a cohomology class**, $[\Phi] \in H^3(Y, \mathbb{R})$ — i.e. $\Phi = \Phi_0 + d\Gamma$ for some fixed closed Φ_0 . So $d\Phi = 0$ of course; and the effect of variation of Γ is

$$\delta V_7(\Phi) = \frac{7}{3} \int_Y *_{\Phi} \Phi \wedge d(\delta\Gamma),$$

so $\delta V_7(\Phi) = 0 \Rightarrow d *_{\Phi} \Phi = 0$.

Hitchin's G_2 volume functional and topological M-theory

So

$$V_7(\Phi) = \int_Y \Phi \wedge *_\Phi \Phi$$

generates G_2 holonomy metrics at its critical points. In this sense it is a candidate action for **topological M-theory**.

By analogy with physical M-theory, one would expect that topological M-theory on $X \times S^1$ should be related to topological string theory on X . Indeed, this V_7 can be connected to the A and B models.

Hitchin's G_2 volume functional and topological M-theory

Namely, letting t be the coordinate along S^1 in $Y = X \times S^1$, and splitting

$$\Phi = kdt + \rho$$

one gets (assuming the constraints $k \wedge \rho = 0$, $2V_S(\sigma) - V_H(\rho) = 0$)

$$*_\Phi \Phi = \hat{\rho}dt + \sigma$$

which implies

$$V_7(\Phi) = 2V_H(\rho) + 3V_S(\sigma)$$

So topological M-theory seems to reduce to the sum of the A and B models at least in this formal sense.

Hamiltonian reduction

Another perspective on this relation: consider **canonical quantization** of topological M-theory on $X \times \mathbb{R}$. The phase space is then $\Omega_{exact}^3(X, \mathbb{R}) \times \Omega_{exact}^4(X, \mathbb{R})$ (omitting zero modes), with the symplectic pairing determined by

$$\langle \delta\rho, \delta\sigma \rangle = \int \delta\rho \wedge \frac{1}{d} \delta\sigma \quad (1)$$

and the Hamiltonian

$$H = 2V_S(\sigma) - V_H(\rho). \quad (2)$$

The conditions $k \wedge \rho = 0$, $2V_S(\sigma) - V_H(\rho) = 0$ then (should) show up as the diffeomorphism and Hamiltonian constraints, as usual for quantization of a diffeomorphism invariant theory.

The A model and B model appear as **conjugate** degrees of freedom!

Open questions

There are many **open questions**:

- ▶ How does topological M-theory embed into the physical string/M-theory (what quantities does it compute?)
- ▶ Can it be used to give a nonperturbative definition of the topological string? The splitting between A and B models is not covariant in 7 dimensions — does this mean the A and B models have to be mixed together nonperturbatively? How should the 6-dimensional couplings g_A , g_B be identified?
- ▶ What is the meaning of the fact that the A and B model appear as conjugate variables? (S-duality?) [Nekrasov-Ooguri-Vafa]
- ▶ Should the theory be augmented to one which describes generalized G_2 structures? [Pestun-Witten, Hitchin, Witt]
- ▶ Is there a lift to 8 dimensions? [Anguelova-de Medeiros-Sinkovics]