Introduction to Supergravity I

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Supergravity is the gauge theory of supersymmetry, much as Yang-Mills theory is a gauge theory of Lie group symmetry and Einstein's theory (General Relativity) is a gauge theory of diffeomorphisms. In this lecture, I discuss supergravity in 3 dimensions. I motivate supergravity by the analogy to Yang-Mills theory and General Relativity.

Let's start with a familiar theory, Yang-Mills theory, and understand it in the spirit of Weinberg's theorem. Consider 3 $U(1)$ connections A, with a gauge transformation

$$i_{\vec{\lambda}\cdot\vec{G}_{U(1)^3}}\delta\vec{A} = d\vec{\lambda}$$

and an action

$$\int |d\vec{A}|^2.$$

(Here "G" means the generator of the group.) This action has a rigid $O(3)$ symmetry

$$i_{\vec{\ell}\cdot\vec{G}_{O(n)}}\delta\vec{A} = \vec{A}\times\vec{\ell}, \quad d\vec{\ell} = 0.$$

We introduce interactions that do not increase the number of derivatives into the theory by promoting the rigid $O(3)$ symmetry to a gauge symmetry. This combines the above variations:

$$i_{\vec{\lambda}\cdot\vec{G}_{O(3)}}\delta\vec{A} = d\vec{\lambda} + \vec{A}\times\vec{\lambda}.$$

The action is

$$\int |\vec{F}|^2$$

where $\vec{F} = d\vec{A} + \vec{A}\times\vec{A}$.

To derive supergravity in an analogous way, we need two ingredients: the analog of the $U(1)$ gauge field, and the analog of the $O(3)$ rigid symmetry. The former is a free Rarita–Schwinger field ψ, a spinor-valued 1-form in three (Minkowski) dimensions. The variation is $i_{\bar{\epsilon}Q}\delta\psi = d\epsilon$. (The notation $\bar{\epsilon}Q$ denotes the spinorial inner product of ϵ and Q.) In three dimensions, the gauge invariant action takes a very simple form:

$$I_{\rm RS} = \int \bar{\psi} \wedge d\psi.$$

This describes a free spin 3/2 field. The action $I_{\rm RS}$ is invariant up to a total derivative term:

$$i_{\bar{\epsilon}Q}\delta I_{RS} = \int d\bar{\epsilon} \wedge d\psi = \int d(\bar{\epsilon} d\psi).$$

The analog of the rigid $O(3)$ symmetry is rigid supersymmetry. As other lecturers have explained, supersymmetry relates fields that differ by spin 1/2. For the spin 3/2 field, this leaves two options: the spin $(1, 3/2)$ and the spin $(2, 3/2)$ multiplets. Both multiplets exist as free multiplets, but only the latter allows us to promote the rigid supersymmetry to a gauge symmetry.[1] The free spin 2 field is the theory of linearized gravity, to which we now turn.

To study supergravity, it is convenient to work with the Cartan formalism. Instead of a metric, we introduce frames $\tilde{e}_a$ and coframes e^a, $a = 1, \dots, d$ with $g = e^a \otimes e^b \eta_{ab}$ and $\langle e^a, e^b \rangle = \eta^{ab}$. The frames and coframes obey $i_{\tilde{e}_a} e^b = \delta_a{}^b$, and $\tilde{e}_a \otimes e^a = Id$, where Id is the identity map on the tangent space.

Instead of affine connections, we introduce Lorentz connection 1-forms which obey

$$De^a = de^a + \omega^a{}_b \wedge e^b = T^a,$$

where T^a is the torsion 2-form. This can be solved for ω:

$$\omega^a{}_b = -i_{\tilde{e}_b} de^a - K^a{}_b, \quad K^a{}_b \wedge e^b \equiv T^a,$$

where K is the contortion 1-form, and like the connection ω obeys $K^a{}_b \eta_{ac} = -K^a{}_c \eta_{ab}$. The connection has a curvature $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$.

The Einstein action $I_{\rm GR}$ is proportional to

$$\int \epsilon^{a_1 \dots a_{d-2} bc} e_{a_1} \wedge \dots \wedge e_{a_{d-2}} \wedge R_{bc}(\omega(e, T)),$$

where $\epsilon^{a_1 \dots a_d}$ is the alternating symbol. In $d = 3$ this simplifies to

$$\int \epsilon^{abc} e_a \wedge R_{bc}.$$

To linearize, we write $e^a = e^a_{(0)} + e^a_{(1)}$ with $de^a_{(0)} = 0$ and $T^a = 0$; we obtain the linearized Einstein action

$$I_{\text{spin-2}} = \int (e^a_{(1)}, \mathcal{O}_{ab} e^b_{(1)}),$$

[1] The underlying reason is the supersymmetry algebra: the commutator of two supersymmetry transformations is a translation. Gauged translations are diffeomorphisms, with the metric as a gauge field, and hence supergravity must involve a spin 2 field.

where $\mathcal{O}$ is the (flat space) Lichnerowicz operator. This action is invariant under gauge translations

$$i_{\xi\cdot P}e^a_{(1)} = d\xi^a \ , \quad \xi^a \equiv i_\xi e^a_{(0)},$$

and rigid Lorentz rotations:

$$i_{\lambda\cdot J}e^a_{(1)} = \lambda^a{}_b e^b_{(1)} \ , \quad d\lambda^a{}_b = 0.$$

The combined action $I_{\rm RS} + I_{\text{spin-2}}$ has in addition a rigid symmetry

$$i_{\bar\eta Q_{\rm FLAT}} e^a_{(1)} = \frac{1}{2}\bar\eta\gamma^a\psi$$

$$i_{\bar\eta Q_{\rm FLAT}}\psi = \omega_{(1)}(J)\eta = -\frac{1}{2}\omega^{ab}_{(1)}\sigma_{ab}\eta \ , \quad d\eta = 0.$$

The generator of Lorentz transformations on spin 1/2 is

$$\sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b]$$

with $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$.

Now we get the three dimensional supergravity action

$$I_{\rm SG} = \int \epsilon^{abc} e_a \wedge R_{bc} + \bar\psi \wedge D\psi$$

where $D\psi = d\psi + \omega(J)\psi = d\psi - \frac{1}{2}\omega^{ab}\sigma_{ab}\psi$. This is symmetric under gauge supersymmetry transformations that combine the rigid supersymmetry transformations with the Rarita–Schwinger gauge symmetry:

$$i_{\bar\epsilon Q} e^a = \frac{1}{2}\bar\epsilon\gamma^a\psi$$

$$i_{\bar\epsilon Q}\psi = D\epsilon.$$

(Note: there is a form of $I_{\rm RS}$ which looks very similar to the $I_{\rm GR}$ which we wrote down, namely, $I_{\rm RS}$ proportional to

$$\int \epsilon^{a_1,\dots,a_{d-3}bcd} e_{a_1} \wedge \cdots \wedge e_{a_{d-3}} \wedge \bar\psi\gamma_b\gamma_c\gamma_d \wedge D\psi.$$

To prove invariance of the action, we introduce the Palatini formalism. We assume that the connection is an independent field, that is, for the moment we have $De^a \neq T^a$, and we extremize the action with respect to variations of the connection only:

$$\delta_\omega I_{\rm GR} = \int \delta\omega_{bc} \wedge \epsilon^{abc} De_a$$

$$\delta_\omega I_{\rm RS} = \int \frac{1}{2}\delta\omega_{bc} \wedge \bar\psi\sigma^{bc} \wedge \psi = \int \frac{1}{4}\delta\omega_{bc} \wedge \epsilon^{abc}\bar\psi\gamma_a \wedge \psi$$

Then $\delta_\omega I_{\rm SG} = 0$ implies

$$De^a = -\frac{1}{4}\bar{\psi}\gamma^a \wedge \psi.$$

Hence we obtain a spin connection that preserves the frames and has torsion $T^a = -\frac{1}{4}\bar{\psi}\gamma^a \wedge \psi$.

Though we do not actually need it in three dimensions, it is useful to explain a simple notion whimsically called the 1.5 order formalism. We consider a system with an auxiliary field A (in the supergravity example, this is the connection ω). An auxiliary field is one whose equations of motion admit a single solution; thus it enters the action with an algebraic quadratic term and an arbitrary linear term. We call the action with the auxiliary field a first-order action:

$$I_1[A, \phi] = \int \frac{1}{2}(A, A)_\phi + (A, f(\phi))_\phi + \mathcal{L}(\phi)$$

Eliminating the auxiliary field by its equation of motion (by extremizing the first-order action with respect to variations of the auxiliary field), we obtain a second-order action

$$I_2[\phi] = I_1[A(\phi), \phi] = \int -\frac{1}{2}(f(\phi), f(\phi))_\phi + \mathcal{L}_\phi(\phi).$$

If we are interested in the invariance of the second-order action, we may write,

$$i_G \delta I_2 = i_G \delta\phi \left(\frac{\delta I_1}{\delta \phi} + \frac{\partial A}{\partial \phi}\frac{\delta I_1}{\delta A} \right).$$

But $\frac{\delta I_1}{\delta A} = 0$ is the condition that defines the second-order action. Then $i_G dI_2 = 0$ if and only if

$$i_G \delta\phi \frac{\delta I_1}{\delta \phi} = 0.$$

In supergravity, the connection 2-form ω is the auxiliary field, and hence we have

$$i_{\bar{\epsilon}Q} I_{\rm SG} = \int \epsilon^{abc}\frac{1}{2}\bar{\epsilon}\gamma^a\psi \wedge R_{bc} + D\bar{\epsilon} \wedge D\psi + \bar{\psi} \wedge D^2\epsilon.$$

The last two terms can be rewritten as

$$2D^2\bar{\epsilon} \wedge \psi - d(D\bar{\epsilon} \wedge \psi).$$

However, the definition of the curvature tensor implies

$$2D^2\bar{\epsilon} = -\bar{\epsilon}\sigma^{ab}R_{ab} = -\frac{1}{2}\epsilon^{abc}\bar{\epsilon}\gamma_c R_{ab}$$

and hence the action is invariant.

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