Introduction to Supergravity II

Martin Roček

July 19, 2001

1 Three dimensions

Review: Last lecture, I tried to motivate and explain how supergravity is the gauge theory of supersymmetry. We had a Rarita–Schwinger field

$$\psi \in \Gamma(T^* \otimes S)$$

and the supergravity action

$$I_{\mathrm{SG}} = \int \epsilon^{abc} e_a \wedge R_{bc}(\omega) + \bar{\psi} \wedge D(\omega)\psi.$$

(This simple form of the action is special to three dimensions.) The Lorentz symmetries are

$$i_{\lambda \cdot J}\delta e^a = \lambda^a{}_b e^b$$

$$i_{\lambda \cdot J}\delta \omega^a{}_b = D\lambda^a{}_b \equiv d\lambda^a{}_b + \omega^a{}_c \lambda^c{}_b.$$

$$i_{\lambda \cdot J}\delta \psi = -\frac{1}{2}\lambda^{ab}\sigma_{ab}\psi.$$

where $a, b, \cdots = 1, 2, 3$. The diffeomorphism invariance is

$$i_{\xi \cdot P}\delta e^a = \mathcal{L}_\xi e^a$$

and similarly for ω and ψ. The supersymmetry transformations are

$$i_{\bar{\epsilon}Q}\delta e^a = \frac{1}{2}\bar{\epsilon}\gamma^a\psi$$

$$i_{\bar{\epsilon}Q}\delta\psi = D\epsilon \equiv d\epsilon - \frac{1}{2}\omega^{ab}\sigma_{ab}\epsilon.$$

$$i_{\bar{\epsilon}Q}\delta\omega = 0.$$

These imply that $i_{\bar{\epsilon}Q}\delta I_{\mathrm{SG}} = 0$.

Varying the action to obtain field equations (equations of motion):

$$\frac{\delta}{\delta\omega}: \quad De^a = -\frac{1}{4}\bar{\psi}\gamma^a \wedge \psi \qquad \Longrightarrow \omega(e,\psi)$$

$$\frac{\delta}{\delta e}: \quad R_{ab}(\omega) = 0.$$

$$\frac{\delta}{\delta\psi}: \quad D(\omega)\psi = 0$$

Supersymmetry rotates the equations of motion, e.g.

$$i_{\bar{\epsilon}Q}\delta(D(\omega)\psi) = D^2\epsilon = -\frac{1}{2}R_{ab}\sigma^{ab}\epsilon$$

Remark: A closely related observation follows from applying the covariant exterior derivative D to the equation of motion of ψ (the Rarita-Schwinger equation): $0 = D^2\psi$ implies $R_{ab}\sigma^{ab} \wedge \psi = 0$, which in turn implies $R_{ab} = 0$ for generic ψ. (Recall $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$.) In this sense, the Einstein equation is the integrability condition for the existence of solutions to the Rarita-Schwinger equation.

What is the supersymmetry algebra? It is easy to see that

$$[i_{\bar{\epsilon}_1 Q}\delta, i_{\bar{\epsilon}_2 Q}\delta]e^a = \frac{1}{2}(\bar{\epsilon}_2\gamma^a D\epsilon_1 - \bar{\epsilon}_1\gamma^a D\epsilon_2) = D(\frac{1}{2}\bar{\epsilon}_2\gamma^a\epsilon_1)$$

This can be rewritten as

$$= \mathcal{L}_\xi e^a + \lambda^a{}_b e^b + \frac{1}{2}\bar{\eta}\gamma^a\psi - i_\xi(De^a + \frac{1}{4}\bar{\psi}\gamma^a \wedge \psi).$$

where $\xi = \frac{1}{2}\bar{\epsilon}_2\gamma^a\tilde{e}_a$, *i.e.*, $\xi^a = i_\xi e^a = \frac{1}{2}\bar{\epsilon}_2\gamma^a\epsilon_1$, and $\lambda^a{}_b = -i_\xi\omega^a{}_b$, $\eta = -i_\xi\psi$. Note that the last term is an equation of motion. This pattern continues for the other fields:

$$[\ ,\ ]\omega = \text{stuff} - i_\xi R^a{}_b$$

$$[\ ,\ ]\psi = \text{stuff} - i_\xi D\psi.$$

Thus the algebra of supersymmetry transformations closes on Lorentz transformations, diffeomorphisms, and sypersymmetries modulo the equations of motion. Physicists say that the algebra closes on-shell. This means that the algebra closes on solutions to the equations of motion, but gets extra generators on arbitrary field configurations. In second-order formalism, *i.e.*, when we impose the ω equation of motion and express it as $\omega(e^a,\psi)$, the algebra closes off-shell on e^a but only on-shell on ψ. In three dimensions, one may find auxiliary fields such that the algebra closes off-shell, but in higher than six dimensions, it is not known how to do this in a satisfactory way, *i.e.*, so that one can write an invariant action. Why is off-shell closure of the algebra an interesting property? In most circumstances, it allows a realization of the algebra that is sufficiently linear, *e.g.*, in superspace, that a sum of invariant actions is automatically invariant.

2 Four dimensions

The action is

$$I_{\mathrm{SG}} = \int \epsilon^{abcd} e_a \wedge e^b \wedge R_{cd}(\omega(e,\psi)) + \epsilon^{abcd} e^a \wedge \bar{\psi}\gamma_{bcd} \wedge D(\omega(e,\psi))\psi.$$

We are in second order formalism, that is, the connection $\omega(e,\psi)$ is determined by $De^a = -\frac{1}{4}\bar{\psi}\gamma^a \wedge \psi$; note, however, that this is compatible (and indeed follows from) with $\delta I/\delta\omega = 0$.

In addtion to four dimensional Lorentz and diffeomorphisms, this action is invariant under:

$$i_{\bar{\epsilon}Q}\delta e^a = \frac{1}{2}\bar{\epsilon}\gamma^a\psi$$
$$i_{\bar{\epsilon}Q}\delta\psi = D\epsilon$$
$$i_{\bar{\epsilon}Q}\delta\omega^a{}_b = \frac{\partial\omega^a_b}{\partial e^c} i_{\bar{\epsilon}Q}\delta e^c + \frac{\partial\omega^a_b}{\partial\psi} i_{\bar{\epsilon}Q}\delta\psi.$$

A first-order formalism is not convenient; we use 1.5-order formalism, which I'll try to explain in a simpler way than last lecture (it really is a triviality): The supersymmetry variation of the action is

$$i_{\bar{\epsilon}Q}\delta I_{\mathrm{SG}} = \frac{\partial I}{\partial e^a} i_{\bar{\epsilon}Q}\delta e^a + \frac{\partial I}{\partial\psi} i_{\bar{\epsilon}Q}\delta\psi,$$

where $\frac{\delta I}{\delta\omega} i_{\bar{\epsilon}Q}\delta\omega = 0$ is automatic because we use $\frac{\delta I}{\delta\omega} = 0$. This is all the 1.5-order formalism means.

The transformations are:

$$\begin{aligned} i_{\bar{\epsilon}Q}\delta I_{\mathrm{SG}} = {} & \epsilon^{abcd}\bar{\epsilon}\gamma_a\psi \wedge e_b \wedge R_{cd} + \frac{1}{2}\epsilon^{abcd}\bar{\epsilon}\gamma_a\psi \wedge \bar{\psi}\gamma_{bcd} \wedge D\psi \\ & + \epsilon^{abcd} e_a \wedge D\bar{\epsilon}\gamma_{bcd} \wedge D\psi + \epsilon^{abcd} e_a \wedge \bar{\psi}\gamma_{bcd} \wedge D^2\epsilon. \end{aligned}$$

We may integrate the third term by parts; dropping the boundary term, it combines with the fourth term to give

$$-\frac{1}{4}\epsilon^{abcd}\bar{\psi}\gamma_a \wedge \psi \wedge \bar{\epsilon}\gamma_{bcd} \wedge D\psi + 2\epsilon^{abcd} e_a \wedge \bar{\psi}\gamma_{bcd} \wedge D^2\epsilon.$$

Using the definition of the Riemann tensor, we find $D^2\psi = -\frac{1}{2}\sigma^{ab}R_{ab} \wedge \psi$, and we can rewrite the last term. Finally using Clifford algebra identities that follow from the basic relation $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$ to reduce products of γ matrices as well as Fierz identities to rearrange the spinor contractions, we can show that these terms all cancel.

The equations of motion that follow from extremizing the action are:

$$\frac{\delta}{\delta e}: \quad \epsilon^{abcd} e_a \wedge R_{cd} + \frac{1}{2}\epsilon^{abcd}\bar{\psi}\gamma_{bcd} \wedge D\psi = 0$$

$$\frac{\delta}{\delta\psi}: \quad \epsilon^{abcd} e_a \wedge \gamma_{bcd} D\psi = 0$$

(with $\omega(e,\psi)$ defined by $De^a + \frac{1}{4}\bar{\psi}\gamma^a \wedge \psi = 0$). The first equation is equivalent to Einstein's equation with the stress-tensor of the Rarita-Schwinger field ψ.

As in three dimensions, equations of motion rotate into equations of motion under supersymmetry. We also find that the consistency of the theory requires torsion:

$$D(\epsilon^{abcd} e_a \gamma_{bcd} \wedge D\psi) = 0$$

implies constraints on the Ricci tensor that are consistent with Einstein's equation only if there is torsion. As in three dimensions, the Einstein equation can be regarded as the integrability condition for the existence of solutions to the Rarita-Schwinger equation.

3 Supergravity and superspace

Let us now return to three dimensions, following a geometric approach. We have a 3-dimensional ordinary manifold with 2-dimensional fuzz: $\mathcal{SM}^{3|2}$. There are local coordinates $x^{\mu\nu}$ and θ^μ for $\mu = +,-$. We don't use a metric on superspace, it is not the right notion.

We introduce bosonic vector fields

$$E_{\alpha\beta} = E_{\alpha\beta}{}^M D_M$$

and fermionic vector fields

$$E_\alpha = E_\alpha{}^M D_M$$

which can be put together as E_A. The bosonic one is a section of $\Gamma(\mathrm{Sym}^2\, S_{\mathcal{M}} \otimes T(\mathcal{M}))$ and the fermionic one is a section of $\Gamma(\Pi S_{\mathcal{M}} \otimes T(\mathcal{M}))$, where S is just the usual $SL(2,\mathbb{R})$ spin bundle of the ordinary manifold in three dimensions.

It is convenient to formulate the derivatives as follows:

$$D_{\mu\nu} = \frac{\partial}{\partial x^{\mu\nu}}$$

$$D_\mu = \frac{\partial}{\partial \theta^\mu} + i\theta^\nu \frac{\partial}{\partial x^{\mu\nu}}.$$

All quantities that we consider take values in $\mathbb{R}$ tensored with an arbitrarily large Grassmann algebra. John Morgan and Dan Freed have given a functorial description of this.

***

The natural continuation of this subject appears in the lectures of S.J. Gates. See also Chapter 2 of *Superspace* by Gates, Grisaru, Roček, and Siegel, which has been copied and is available.

Once again, I am delighted to thank Dave Morrison for writing the first draft of these notes.