2. Lecture 2

2.1. Some Facts about Superconformal Field Theories

We will not have the time for a real study of Superconformal Field Theory, but I would like to digress a bit here and survey some points that will be relevant to our discussion.

In general, the behavior of a quantum field theory in the extreme low-energy limit, when nontrivial, is determined by a fixed point of the Renormalization Group flow, i.e. by a theory with no intrinsically determined scale - a *conformal field theory.* In such theories the Poincaré symmetry is extended to an invariance under the action of the conformal group of coordinate transformation preserving the (Minkowski or Euclidean) metric up to a local scale factor. In two dimensions, the algebra of such transformations is infinite dimensional. This fact is related to the splitting of the light cone mentioned above, and is manifest in the fact that the transformation

$$z \to f(z) \tag{2.1}$$

is a conformal transformation for any holomorphic function f. The conformal algebra consists of two commuting copies $\mathrm{Vir} \oplus \overline{\mathrm{Vir}}$ of the Virasoro algebra, acting separately on z and $\bar{z}$, with nontrivial brackets

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{m+n,0} \ . \tag{2.2}$$

These generators are the modes of the energy-momentum stress tensor. In particular, we have

$$L_0 = H + P \tag{2.3}$$

for the holomorphic copy and

$$\bar{L}_0 = H - P \tag{2.4}$$

for the antiholomorphic copy of the algebra. c is a central charge of the algebra, normalized so that for a free scalar field we have $c = 1$. The $sl(2,\mathbb{R})$ subalgebra generated by $L_0, L_{\pm 1}$ suffers no central extension and represents the action of the global holomorphic reparametrizations of a two-sphere. The states in the Hilbert space form a highest-weight representation of this algebra; the highest weight states satisfy

$$L_n|\phi\rangle = 0 = \bar{L}_n|\phi\rangle \qquad n > 0 \ . \tag{2.5}$$

We label them by their eigenvalues $(h, \bar{h})$ under L_0 and $\bar{L}_0$. The vacuum state with $L_0|0\rangle = \bar{L}_0|0\rangle = 0$ is assumed unique. In a conformal field theory, there is a one-to-one correspondence between local fields and states in the Hilbert space via

$$\mathcal{O} \leftrightarrow \mathcal{O}(0)|0\rangle \ . \tag{2.6}$$

In a conformal field theory, the local operators (fields) form an associative algebra under the operator product. That is, the series expansion

$$A(z)B(w) = \sum_C \frac{C(w)}{(z-w)^{h_A+h_B-h_C}} \tag{2.7}$$

is in fact convergent in a punctured disk.

In the presence of $N = 2$ SUSY the symmetry is enhanced. As with the Poincaré algebra we here find a symmetry generator for each mode of the supercurrents. In addition, we find that the $U(1)$ R-symmetry current is incorporated into the superconformal algebra. In all, we have the holomorphic symmetry generators

$$\begin{aligned} T(z) &= \sum_n L_n z^{-n-2} && \text{energy-momentum tensor. } L_0 = H + P \\ J(z) &= \sum_n J_n z^{-n-1} && \text{R-symmetry current. } J_0 = R \\ G^\pm(z) &= \sum_n G^\pm_{n\pm a} z^{-(n\pm a)3/2} && \text{supercurrents. } G^\pm_0 = Q^\pm \ . \end{aligned} \tag{2.8}$$

Note the parameter a which determines the moding, hence periodicity, of the fermionic generators. The theories we consider have two sectors, the R(amond) sector for which $a = 0$ and the N(eveu)-S(chwarz) sector for which $a = 1/2$. The nonvanishing brackets here are

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{m+n,0} \\ [J_n, J_m] &= \frac{c}{3}m\delta_{m+n,0} \\ [L_n, J_m] &= -mJ_{m+n} \\ [L_n, G^\pm_{m\pm a}] &= \left(\frac{n}{2}(m \pm a)\right) G^\pm_{m+n\pm a} \\ [J_n, G^\pm_{m\pm a}] &= \pm G^\pm_{m\pm a} \\ \{G^+_{n+a}, G^-_{m-a}\} &= 2L_{m+n} + (n-m+2a)J_{m+n} + \frac{c}{3}\left((n+a)^2 - 1/4\right)\delta_{m+n,0} \ . \end{aligned} \tag{2.9}$$

These generators satisfy the reality conditions

$$\begin{aligned} L_n^\dagger &= L_{-n} \\ J_n^\dagger &= J_{-n} \\ \left(G^{\pm}_{n\pm a}\right)^\dagger &= G^{\mp}_{-n\mp a} \ . \end{aligned} \tag{2.10}$$

There is a second copy of the same algebra acting on the antiholomorphic fields.

The highest-weight states in this case are labeled by their charges under the Cartan subalgebra. In the NS sector this consists of the eigenvalues (q,h) under $J_0 = R$ (for the holomorphic algebra, and $\bar{q}$ under $\bar{J}_0 = L$ for the antiholomorphic algebra) and $L_0 = H+P$ (resp. $\bar{h}$ under $\bar{L}_0 = H-P$). In the R sector we have in addition the zero modes $G_0^\pm = Q^\pm$.

Of particular interest to us are the following states: In the NS sector we have *chiral primary states* (or fields) which satisfy

$$G^+_{-1/2}|\phi\rangle = 0 \ . \tag{2.11}$$

We can now make a computation. For any NS state consider the expression

$$|\langle\phi|\{G^-_{1/2}, G^+_{-1/2}\}|\phi\rangle \ . \tag{2.12}$$

Our reality conditions (2.10) guarantee it is non-negative; for a chiral primary state it vanishes. On the other hand, we have from the algebra (2.9)

$$\{G^-_{1/2}, G^+_{-1/2}\} = 2L_0 - J_0 \ . \tag{2.13}$$

Thus we see that for any eigenstate we have $h \geq q/2$, while chiral primaries saturate this.We can also show that for such a state $h \leq c/6$. Under the operator product algebra, these fields close among themselves to form the *chiral primary ring* of the theory. We also have *antichiral primary states* satisfying

$$G^-_{-1/2}|\phi\rangle = 0 \ , \tag{2.14}$$

hence $h = -q/2$, forming the *antichiral primary ring.* When we consider both the holomorphic and antiholomorphic degrees of freedom we thus have four rings, labeled (c,c), (a,c), (c,a), and (a,a). Complex conjugation relates the latter two to the former two.

Chiral (or antichiral) primary fields with $|q| = |\bar{q}| = 1$ are of special interest. Associated to them are *exactly marginal operators*, i.e. operators which can be added to the Lagrangian deforming the theory without spoiling the superconformal invariance. For example, for a (c,c) field h with charges $(1,1)$, the operator $C = G^-_{-1/2}\bar{G}^-_{-1/2}h$ is a neutral field of conformal weight $(1,1)$. It variation under the algebra is a total derivative. In a general CFT, marginal operators exist which are not exactly marginal, in other words there are obstructed deformations. For the marginal operators associated as above to chiral primary fields in an $N=2$ superconformal theory the obstruction vanishes.

In the R sector the states we wish to consider are the *supersymmetric vacua* which satisfy

$$G^+_0|\phi\rangle = G^-_0|\phi\rangle = 0 \; . \tag{2.15}$$

The algebra (2.9) then shows that these states have weight $h=0$.

Let us consider now a few examples of superconformal field theories to demonstrate these properties.

The first example is the free theory of a single chiral multiplet discussed in the first lecture. This rather trivial example is important because it forms the basis of much of our physical intuition. The Lagrangian in component notation is (after Wick rotation)

$$L = \partial\phi\bar{\partial}\bar{\phi} + i(\bar{\psi}^-\partial\psi^- + \bar{\psi}^+\bar{\partial}\psi^+) \; , \tag{2.16}$$

where $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$. It is a good exercise to show that the currents

$$\begin{aligned} T(z) &= \partial\phi\partial\bar{\phi} + \tfrac{1}{2}(\bar{\psi}^+\partial\psi^+ + \psi^+\partial\bar{\psi}^+) \\ J(z) &= \frac{1}{4}\bar{\psi}^+\psi^+ \\ G^+(z) &= \frac{1}{2}\bar{\psi}^+\partial\phi \\ G^-(z) &= \frac{1}{2}\psi^+\partial\bar{\phi} \end{aligned} \tag{2.17}$$

satisfy the algebra (2.9) with central charge $c=3$. To find the chiral rings check that ψ^+ has charges $(1,0)$ under (R,L) and conformal weights $(h,\bar{h}) = (1/2,0)$. It is thus a chiral primary. The (c,c) ring here is spanned by $(1,\psi^+,\psi^-,\psi^+\psi^-)$ and the (a,c) ring by $(1,\bar{\psi}^+,\psi^-,\bar{\psi}^+\psi^-)$. Note that the conformal weights of the highest element in each ring saturate our inequality $h \le c/6$. Note also the absence of marginal chiral primary fields.

We would like to extend this example to study the superconformal field theory obtained from the nonlinear sigma model with a Calabi-Yau target space. The analysis here is more difficult, and we will reproduce it in a modified version of the theory in which things simplify, but let me state the results. The central charge of the theory is $c = 3n$

where $n = \dim_{\mathbb{C}} M$. The chiral rings in this case are isomorphic as vector spaces to sums of particular cohomology groups of M. The (c, c) ring has states of (R, L) charges (p, p) for $p = 0, \ldots, n$ corresponding to elements of $H^p(\wedge^p TM)$; the (a, c) ring has states of charges $(p, -p)$ corresponding to elements of $H^p(\wedge^p T^*M)$. Marginal chiral primaries thus correspond to elements of $H^1(TM)$ and of $H^1(T^*M)$. The geometric interpretation of the ring structures will be extremely interesting to us and will be discussed below.

Let us discuss one more important example. Consider the theory of a single chiral multiplet with the free kinetic term modified by an F-term interaction determined by the superpotential

$$W(\phi) = \frac{1}{k+2}\phi^{k+2} \ . \tag{2.18}$$

The ground states of this theory are concentrated near the critical point of W at $\phi = 0$. This is *not* a conformally invariant theory. Its extreme low- energy physics, however, turns out to be determined by a nontrivial superconformal field theory. It is instructive to study the R-symmetry in this case. The naïe R-symmetries are broken by W; however, the quasihomogeneity allows us to construct modified R-symmetries, under which Φ has charges $(\frac{2}{k+2}, \frac{2}{k+2})$. The existence of an unbroken R-symmetry is an indication that the extreme low-energy (infrared, or IR) behavior might be described by a nontrivial SCFT. It is a property of supersymmetric models that the superpotential is not modified by RG flow (except by field redefinitions). This is *not* true for the kinetic term, which we expect to undergo severe renormalization. This model was studied by Vafa and Warner, who found that the IR fixed point of the RG flow is indeed an $N = 2$ SCFT, with central charge $c = \frac{3k}{k+2}$. The (c, c) ring is isomprphic (as a ring!) the Jacobian ring of W, i.e.

$$\frac{\mathbb{C}[\phi]}{\partial W(\phi)} \tag{2.19}$$

in general. In our case this is spanned by $(1, \Phi, \Phi^2, \ldots, \Phi^k)$. The (a, c) ring was shown to be trivial, spanned by 1.

Notice that in this case the central charge c is *less* than what we found for one free chiral multiplet. This is important. For such values of the central charge the representation theory of the algebra (2.9) simplifies dramatically. In fact, these theories are *exactly solved* in the sense that the spectrum of states as well as all correlation functions can be explicitly computed from the algebra (2.9) directly. These *minimal models* played an important part in the discovery of mirror symmetry, as we shall discuss below.

2.2. Some facts about Calabi-Yau spaces

Since we are going to be studying Calabi-Yau spaces, let me collect some facts about these as well.

The most directly calculable topological invariants of a Kähler manifold are the dimensions of the Dolbeault groups $h^{(p,q)}$. The vanishing of the first Chern class, or equivalently the triviality of the canonical line bundle, means these spaces have a nowhere vanishing holomorphic n-form, determined up to a constant and usually denoted Ω. Further, if the holonomy is exactly $SU(n)$ (which is our definition of a Calabi-Yau space) then $h^{p,0} = 0$ for $1 \leq p < n$. The Hodge diamond in which these invariants are often written thus simplifies. For $n = 2$ we have

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & h^{1,1} & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{2.20}$$

In fact, there is a unique compact Calabi-Yau manifold of dimension two, the K3 surface with $h^{1,1} = 20$. For $n = 3$ we find

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{2,2} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{2.21}$$

(where of course $h^{p,q} = h^{n-p,n-q}$) and there are many (though conjecturally a finite number) of topological types.

Let us consider the moduli space of Ricci-flat metrics on a Calabi-Yau space of a given topological type. As remarked above, Yau's theorem essentially supplies the answer. A Ricci-flat metric is uniquely specified by a choice of complex structure and a choice of the Kähler class. Deformations of the complex structure are parameterized to first order by $H^1(TM)$. In the case of a Calabi-Yau space, a theorem of Bogomolov asserts that these are unobstructed. Deformations of the Kähler class consist of a cone in the vector space $H^1_{\mathbb{R}}(T^*M)$. The cone is determined by the fact that for any algebraic cycle C of dimension k in M, the Kähler class must satisfy

$$\int_C J^k > 0 \ . \tag{2.22}$$

The B-field is a closed two-form. In fact, the action clearly depends only upon the cohomology class of B; furthermore, integer shifts of B change the action by an integer, leaving the path integral measure $e^{2\pi i S}$ unchanged. Thus, in the case $n = 3$ at least, B takes values naturally in $H^{1,1}(M, \mathbb{R})/H^{1,1}(M, \mathbb{Z})$. We can use this to complexify the Kähler cone, effectively including the torus of B-fields over each point in the cone. These statements are in agreement with our discussion of marginal operators in the nonlinear sigma model in the previous subsection.

How do we construct such examples? Kähler manifolds are directly obtained as analytic submanifolds of other Kähler manifolds. Starting with everyone's favorite example, projective space, we can form a Calabi-Yau threefold as a hypersurface M in $\mathbb{CP}^4$. This will be the solution of $P = 0$ for some homogeneous polynomial P of degree q in five complex variables. For what value of q will this have vanishing first Chern class? Using the splitting principle we have

$$c(\mathbb{CP}^4) = (1 + J)^5 \ , \tag{2.23}$$

where J is the Kähler class of $\mathbb{CP}^4$. On the other hand, we have $T(\mathbb{CP}^4)|_M = T(M) \oplus N(M)$. The line bundle $N(M)$ has Chern form $(1 + qJ)$. We formally write

$$c(M) = \frac{(1 + J)^5}{1 + qJ} = 1 + (5 - q)J + \cdots \tag{2.24}$$

Thus to get a Calabi-Yau hypersurface we should choose P to be a quintic polynomial. Plugging in and expanding further we see that $c_3(M) = -40J^3$ in this case. Integrating over M (and noting that the integral of J^3 is dual to the intersection with three coordinate hypersurfaces so equal to 5) we find $\chi(M) = -200$. Indeed, this space has $h^{1,1} = 1$ (a unique Kähler form up to scale) and $h^{2,1} = 101$. These latter can be described as deformations of the defining polynomial (126 of these) modulo the 25 dimensional group $Gl(5, \mathbb{C})$ of holomorphic reparameterizations of $\mathbb{CP}^4$.

We can generalize this to weighted projective spaces. These are variations on $\mathbb{CP}^4$ given by a modified action of $\mathbb{C}^*$ as

$$\mathbb{WCP}^4_{w_1,\ldots,w_5} = \mathbb{C}^5 - \{0\}/\mathbb{C}^* \ , \tag{2.25}$$

where the action is via $\lambda : (x_1, \ldots, x_5) \to (\lambda^{w_1} x_1, \ldots, \lambda^{w_5} x_5)$. It is immediately clear that these are not manifolds, since the action is not free. It turns out however, that the nonlinear sigma model on spaces with such quotient singularities is well-defined and nonsingular. We will discuss this later. To construct a Calabi-Yau hypersurface in such a space we note that $c(\mathbb{WCP}) = \prod(1 + w_I J)$. $c(NM)$ for M the vanishing locus of a polynomial of degree q (ie $P(\lambda(x)) = \lambda^q P(x))$ is $1 + qJ$, and as before we obtain a Calabi-Yau hypersurface if $q = \sum w_i$. We will study a much richer generalization later on.

We can now describe the first explicit examples of mirror symmetry and the way in which they were discovered. The details of the construction are technical and out of our way, but we can describe the results. Recall that we discussed the exactly solved minimal model theories mentioned above, with central charges $c_k = \frac{3k}{k+2}$. We can form tensor products of these in various ways to obtain a total central charge of $c = 9$ as appropriate for a nonlinear sigma model on a Calabi-Yau threefold. The total L and R charges of the states, however, will not be integers as is true for the sigma model. A construction fue to Gepner allows us to modify the model, essentially gauging some discrete symmetries, to obtain a model with integral charges that can in principle be a point in the moduli space of nonlinear sigma models on some target space. We now have strong evidence that this is indeed the case, and this will be discussed at length in coming lectures. For example, by taking five copies of the model with $k = 3$ we find a model with $c = 9$. Gepner's construction in this case amounts to gauging a $\mathbb{Z}_5$ discrete symmetry of the model. The resulting SCFT, which is exactly solved, describes a point in the moduli space of nonlinear sigma models on the quintic hypersurface in $\mathbb{CP}^4$. Note that the superpotential of the model (before the quotient) is

$$W = \frac{1}{5}\sum_{i=1}^{5} \Phi_i^5 \ . \tag{2.26}$$

This is a quintic polynomial in five variables, and is indeed the polynomial describing this theory. The value of the Kähler class is less clear, and we will discuss it.

The exactly solved minimal models have a property which follows easily from their algebraic solution (and can also be derived more geometrically). The k- th minimal model has a $\mathbb{Z}_{k+2}$ discrete symmetry. If we gauge this, we obtain a quotient model which is isomorphic to the original, except that all fields have their L-charges reversed. A moment's reflection shows that the quotient thus has a nontrivial (a, c) ring spanned by $(1, \hat{\Phi}, \ldots, \hat{\Phi}^k)$ where $\hat{\Phi}$ is the field related to Φ by the isomorphism mentioned. Pulling this property through Gepner's quotient leads to the following prediction in this case: If we consider the quintic hypersurface in $\mathbb{CP}^4$, and choose the complex structure exhibited in (2.26), and then perform a quotient by a discrete $\mathbb{Z}_5^3$ symmetry in the nonlinear sigma model, we will obtain a superconformal field theory isomorphic to the one with which we started, except for the reversal of all L-charges. The crucial point is, that we understand that performing this quotient in the field theory is tantamount to studying the nonlinear sigma model on the associated quotient of the original manifold. This will of course have quotient singularities, but these, as mentioned above, do not render the sigma model singular.

The quotient singularities have small resolutions (preserving the triviality of the canonical bundle) and so this space forms some sort of boundary point in the moduli space of some smooth manifolds. Note that reversing the L-charges has the effect here as well of exchanging the roles of the (c, c) and (a, c) rings. The new space W we have constructed should have $h^{2,1} = 1$ (a one- dimensional space of complex structures) but $h^{1,1} = 101$. In particular, its Euler number ought to be 200. Indeed, this is not too hard to verify. It turns out that the discrete action fixes 10 curves of the form

$$C_{ijk} : x_i^5 + x_j^5 + x_k^5 = 0 \qquad i, j, \text{ distinct} \tag{2.27}$$

invariant under a $\mathbb{Z}_5$ action. These meet in ten fixed points each of which is invariant under a $\mathbb{Z}_5^2$ action. Each curve contains three fixed points and three curves meet at a point. We can use this to show that the Euler number of the quotient is indeed 200. The Hodge diamonds of M and W are related by a reflection about a diagonal axis. This is the source of the term “mirror symmetry”.

A similar construction can be made for any model obtained by Gepner’s method as well as any quotient of such a model, a few hundreds of examples. These are important because in these cases and *only* in these cases, has the exact equivalence of conformal field theories been demonstrated.

For a pair of mirror manifolds, we thus predict the existence of a map from the moduli space of complex structures on the one to the moduli space of Kähler structures on the other, together with an action on the fields in the superconformal field theory, such that the correlation functions are preserved. To get more geometrical information from this, we need to understand how to interpret the correlation functions, and to this we now turn.