3. Lecture 3

3.1. Twisting and Topological Field Theories

The correlation functions of the chiral (and antichiral) primary fields in an $N = 2$ supersymmetric field theory are simpler than those of generic fields. The supersymmetry imposes constraints on their dependence on parameters. There are variants of the theory, due to Witten, that isolate this simple subset of the fields and make the simplifications manifest. We now turn to these. Recall our Lagrangian

$$2\pi\alpha' L = \tfrac{1}{2}(g_{IJ} + iB_{IJ})\partial\phi_I\overline{\partial}\phi^J + ig_{i\bar{\jmath}}\left(\overline{\psi}^{\bar{\jmath}}_{-}D_z\psi^i_{-} + \overline{\psi}^{\bar{\jmath}}_{+}D_{\overline{z}}\psi^i_{+}\right) + R_{i\bar{k}j\bar{l}}\psi^i_{+}\psi^j_{-}\overline{\psi}^{\bar{k}}_{-}\overline{\psi}^{\bar{l}}_{+}\ , \quad (3.1)$$

where $I, J = 1, \ldots, 2d$ are real indices. (I am changing notation and the complex dimension of M is d, to reserve n for another use.) One now constructs two "twisted" variants of the theory by modifying the spins of the odd fields. Physically one can achieve this by coupling the currents for the R and L symmetries to a background gauge connection on Σ determined by the spin connection on Σ. In terms of the symmetry algebra, one chooses to interpret a different subgroup of $U(1)_E \times U(1)_L \times U(1)_R$ as the new rotation generator where $U(1)_E$ is the generator of Euclidean rotations. Explicitly, recall that we had on Σ two spin bundles $S^{\pm}$, and that on a Riemann surface

$$\begin{aligned}\left(S^{+}\right)^{\otimes 2} &= K \\ \left(S^{-}\right)^{\otimes 2} &= \overline{K}\ ,\end{aligned} \quad (3.2)$$

where K is the canonical line bundle of Σ.

We can take the spinors ψ^i_{+} and $\overline{\psi}^{\bar{\imath}}_{+}$ to be sections of $\Pi\Phi^*\left(T^{1,0}M\right)$ and $K \otimes \Pi\Phi^*\left(T^{0,1}M\right)$. Witten calls this a $+$ twist. The Lagrangian is unchanged except that the $D_{\overline{z}}$ operator must now be interpreted as acting on the appropriate bundle. Alternatively, we can make a $-$ twist, making ψ^i_{+} and $\overline{\psi}^{\bar{\imath}}_{+}$ sections of $K \otimes \Pi\Phi^*\left(T^{1,0}M\right)$ and $\Pi\Phi^*\left(T^{0,1}M\right)$ respectively. Similarly, we can twist the left-moving fermions ψ_{-} and $\overline{\psi}_{-}$ in one of two ways. The four possible combinations of twists we obtain are related pairwise by reversing the complex structure of M. Thus we can construct two distinct theories. We call the **A** model the theory obtained by making a $+$ twist of ψ^{+} and a $-$ twist of ψ^{-}. The **B** model is obtained by making $-$ twists on both chiralities. Note that since mirror symmetry reverses the sign of the L-charge, it relates the two types of models. Thus, if M and W are two Calabi-Yau spaces related by mirror symmetry, then $\mathbf{A}(M) = \mathbf{B}(W)$.

What do we gain by performing this twist? On a flat worldsheet, clearly nothing. If K is trivial we have not changed anything. Our symmetry algebra, for example, is unchanged except that we have redefined the generator of Euclidean rotations by adding $\frac{1}{2}(R \pm L)$ so that we label the generators differently. What we achieve is the fact that two of our supercharges – Q_+ and $\overline{Q}_-$ in the **A** model or $\overline{Q}_\pm$ in the **B** model – are worldsheet scalars. This means we have a canonically defined global supersymmetry transformation on any Σ (obtained by setting the parameters of the scalar SUSY transformations to constants). In each model we can form the scalar operator

$$\begin{aligned} Q_A &= Q_+ + \overline{Q}_- \\ Q_B &= \overline{Q}_+ + \overline{Q}_- \ . \end{aligned} \tag{3.3}$$

These satisfy $Q_A^2 = Q_B^2 = 0$ as a result of the algebra.

A scalar nilpotent symmetry like Q is often known as a BRST symmetry, and it allows us to reduce the theory to a much simpler one as follows. If we consider correlation functions of operators $\mathcal{O}_a$ "closed" under Q, i.e. satisfying $\{Q, \mathcal{O}_a\} = 0$, then we find that the insertion into such a correlation function of any operator of the form $A = \{Q, B\}$ leads to a vanishing result, because we can move the insertion of Q through the closed operators to annihilate the vacuum (assumed Q-invariant) on either side. We thus learn that our space of operators may be taken to be the *cohomology* of Q, that is we consider closed operators but we may also identify $\mathcal{O} \sim \mathcal{O} + \{Q, B\}$ for any B. A similar argument holds for the states in Hilbert space. Q-closed operators will have matrix elements invariant under $|\phi\rangle \to |\phi\rangle + Q|\psi\rangle$ for any $|\psi\rangle$, so the Hilbert space can also be taken to be the cohomology of Q.

An important comment here is that the algebra (2.9) now implies some useful invariances of the theory. In particular, in the A model for example we have

$$\{Q^+, G^-(z)\} = 2T'(z) \tag{3.4}$$

to within a constant, where $T'(z) = T(z) - \frac{1}{2}\partial J(z)$ is the modified energy-momentum stress tensor. Thus T' is itself Q-trivial. A similar argument shows that $\overline{T}'$ is Q-trivial; naturally this holds in the **B** model as well. T' determines the coupling of the theory to the metric on Σ, in the sense that

$$\frac{\delta}{\delta\gamma^{ab}}\langle\cdots\rangle = \langle T_{ab}\cdots\rangle \ ,$$

where the correlator is arbitrary. For our correlators the right-hand side vanishes. The correlators are thus completely independent of the choice of a metric on Σ as well as of the positions of the operator insertions. Because of this invariance such theories are called "topological field theories."

We will find further invariances. A deformation of the theory which amounts to modifying the Lagrangian by

$$L \to L + \{Q, \Lambda\}$$

for any Λ will have no effect on our theory. We will shortly see how this affects our models.

Before treating the two types of topological sigma models in detail, there are two more general points I would like to mention. The first is an argument due to Witten that leads to a very simple derivation of many of the interesting results. I am not sure how rigorous this can be made, and most results can be derived without it as I will indicate, but it is an extremely useful tool. Maybe someone here can make it precise? The result in question is the localization of the path integral in a theory with a fermionic symmetry Q onto the fixed-point set of Q- invariant field configurations.

The argument is simple. Consider first a theory with field space $\mathcal{F}$ and a group of (even) symmetries G. If G acts freely on $\mathcal{F}$ then we have a fibration $\mathcal{F} \to \mathcal{F}/G$ and the correlation functions of G-invariant observables using a measure derived from the G-invariant action can be computed as integrals over the base of this, since the integral over the fibers simply produces a factor of the volume of G. Now consider the case of an *odd* symmetry, like our Q. If Q were freely acting we could make a similar argument, except that the volume of the $(0|1)$ supergroup generated by Q vanishes. More explicitly, given a free action of Q, we could introduce an odd collective coordinate on field space such that Q acted as $\frac{\partial}{\partial\theta}$. But then Q- invariant operators are θ independent and we find that our path integral contains the factor

$$\int d\theta = 0 \ .$$

Of course, in real life the action is not free, but the argument shows (by breaking $\mathcal{F}$ into a Q-invariant neighborhood of the fixed locus $\mathcal{F}_0$ and its complement) that the path integral *localizes* onto the fixed-point locus. In the generic case, the measure on $\mathcal{F}_0$ is simply the restriction of the measure on $\mathcal{F}$ weighted by a one-loop determinant of the transverse modes. In both of the models we study, this determinant will in fact be one as a result of boson- fermion cancellations. This argument is extremely useful in reducing path integrals in topological field theories to integrals over more manageable spaces.

The second point concerns the relation between correlation functions in the twisted theories and those in the original conformal theory. The point here is that on a *flat* worldsheet Σ the two are manifestly identical (provided we make the spin bundles trivial in the untwisted model). Consider therefore a worldsheet consisting of an infinite flat cylinder. The path integral on this is determined by a choice of initial and final states. In the twisted model, we choose some representatives $|i\rangle$ and $|f\rangle$ of the relevant Q-cohomology classes (note that these states are chosen in the Hilbert space of the *untwisted* theory). Inserting some Q-invariant operators at points σ_a we compute

$$\langle f| \prod \mathcal{O}_a(\sigma_a)|i\rangle \ . \tag{3.5}$$

We can describe the same correlator after a conformal transformation which turns the cylinder into a sphere. The two ends of the cylinder are replaced by punctures at $\sigma_{i,f}$. As usual, the states $|i\rangle$ and $|f\rangle$ are replaced by the corresponding operators $\mathcal{O}_i$ and $\mathcal{O}_f$ which are some Q-invariant operators. The matrix element (3.5) is the same (since the topological theory is certainly conformally invariant) as the correlator

$$\langle \mathcal{O}_f(\sigma_f)\mathcal{O}_i(\sigma_i) \prod \mathcal{O}_a(\sigma_a)\rangle \ . \tag{3.6}$$

Now, returning for a moment to (3.5) we note that since we can choose K on the cylinder to be trivial, the twisted calculation is equivalent to some matrix element in the untwisted theory. The $\mathcal{O}_a$ correspond to some operators in the untwisted theory. In fact, when we consider the form of Q we see that they correspond precisely to the elements of the appropriate chiral ring of the model ((a,c) in the **A** model and (c,c) in the **B** model).

The states $|i,f\rangle$ also represent some states in the Hilbert space of the untwisted theory. In fact, in choosing the spin bundles to be trivial, we have ensured that fermions satisfy periodic boundary conditions around the cylinder, so these are states in the Ramond sector. In fact, they can be chosen to be Ramond groundstates. In the untwisted theory therefore, the matrix element we are computing is the matrix element of the same operators between the same states, which we now interpret as the matrix element of some chiral primary fields between two RR ground states. From a string theory point of view this is the coefficient of some term in the spacetime superpotential.

Once more we can, if we wish, perform a conformal transformation to the sphere and replace the states by the corresponding operators. In this case, the R states will correspond to fermion vertex operators $V_{i,f}$ (related by spectral flow or spacetime SUSY to the chiral fields $\mathcal{O}_{i,f}$), and so the correlator in the untwisted theory which is numerically equal to (3.6) is

$$\langle V_f(\sigma_f)V_i(\sigma_i)\prod \mathcal{O}_a(\sigma_a)\rangle \ . \tag{3.7}$$

These correlators in the untwisted theory are precisely equal to those we compute in the twisted theory. Of course this is a small minority of the correlation functions of this much richer theory; in particular these correlators enjoy all the invariances of topological correlators independence of the position of the operators, metric on Σ, etc. But this subset of simple correlators are calculated exactly by the twisted theory.

We now turn to a detailed consideration of the two models.

3.2. The A model

In the **A** model we have $2d$ fermionic scalars ψ_+^i and $\overline{\psi}_-^{\bar{\imath}}$, sections of $\Pi\Phi^*\left(T^{1,0}M\right)$ and $\Pi\Phi^*\left(T^{0,1}M\right)$ respectively. We can combine these into a section χ of $\Pi\Phi^*(TM)$. The remaining odd fields will be denoted $\psi_{\bar{z}}^i$, a section of $\overline{K}\otimes\Pi\Phi^*\left(T^{1,0}M\right)$ and $\overline{\psi}_z^{\bar{\imath}}$, a section of $K\otimes\Pi\Phi^*\left(T^{0,1}M\right)$. The transformation properties under our two scalar SUSY generators are

$$\begin{aligned}
\hat{\zeta}\phi^i &= -\sqrt{2}\eta_-\chi^i \\
\hat{\zeta}\phi^{\bar{\imath}} &= \sqrt{2}\overline{\eta}_+\chi^{\bar{\imath}} \\
\hat{\zeta}\chi^I &= 0 \\
\hat{\zeta}\overline{\psi}_z^{\bar{\imath}} &= -i\sqrt{2}\partial\phi^{\bar{\imath}}\eta_- + \sqrt{2}\overline{\eta}_+\chi^{\bar{\jmath}}\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}}\overline{\psi}_z^{\bar{k}} \\
\hat{\zeta}\psi_{\bar{z}}^i &= -i\sqrt{2}\overline{\partial}\phi^i\overline{\eta}_+ + \sqrt{2}\eta_i\chi^j\Gamma^i_{jk}\psi_{\bar{z}}^k \ .
\end{aligned} \tag{3.8}$$

We have eliminated the auxiliary fields, so the algebra $Q_+^2 = \overline{Q}_-^2 = 0$ holds modulo the equations of motion (on-shell). Setting $\eta_- = \overline{\eta}_+$ we find the transformation generated by Q_A. The essential point is that the Lagrangian (3.1) can now be written (modulo the ψ equations of motion) as

$$L = i\{Q,V\} - i\phi^*(K) \ , \tag{3.9}$$

where

$$V = g_{\bar{\imath}j}\left(\overline{\psi}_z^{\bar{\imath}}\overline{\partial}\phi^j + \partial\phi^{\bar{\imath}}\psi_{\bar{z}}^j\right) \tag{3.10}$$

and

$$\phi^*(J) = (B_{i\bar{\jmath}} + ig_{i\bar{\jmath}})\left(\partial\phi^i\overline{\partial}\phi^{\bar{\jmath}} - \overline{\partial}\phi^i\partial\phi^j bar\right) \tag{3.11}$$

is the pullback of the "complexified Kähler class" $K = B + iJ$. This expression places severe restrictions upon the dependence of **A** model correlators on the parameters of the model. Our previous discussion shows that this dependence can come only through the second term in (3.9) since the first term is Q-exact.

This second term itself is quite simple, depending on parameters only through the cohomology class of K and on the fields only through the homotopy class of the map ϕ. Homotopically nontrivial maps are referred to as "world-sheet instantons." In particular, **A** model correlators are independent of the choice of a complex structure on M. The dependence on the Kähler class is restricted to the instanton corrections. In particular, this shows the absence of perturbative corrections to these correlators.

Let us choose a basis for $H^2(M,\mathbb{Z})$, and parameterize our choice of complexified Kähler class by writing

$$K = \sum_a 2\pi t^a e_a \ . \tag{3.12}$$

Then the path-integral measure $e^{-\int_\Sigma L}$ includes a factor

$$e^{i\int_\Sigma \phi^* K} = e^{2\pi i \sum_a n_a t^a} = q^n \ , \tag{3.13}$$

where n_a are integers the instanton number or multidegree of the map ϕ, $q_a = e^{2\pi i t^a}$, and we introduce the notation $q^n \equiv \prod_a q_a^{n_a}$.

The path integral breaks up into disconnected components indexed by n_a. In each component, the t dependence is given by (3.13). Pulling out this factor the contribution is t-independent and can be evaluated in the small-q limit of large $\mathrm{Im}\, t$. In this limit the measure will be concentrated near the critical points of the action. Looking at (3.10) we see that these occur when

$$\partial\phi^{\bar{\imath}} = \overline{\partial}\phi^i = 0 \ , \tag{3.14}$$

i.e. ϕ is a holomorphic map. Note that this is in complete agreement with our fixed-point argument above. Field configurations fixed by Q will have $\chi = \psi = 0$ and ϕ holomorphic. So the path integral in each sector reduces to an integral over the finite-dimensional moduli space of holomorphic maps in the appropriate class.

Let us construct the observables of the theory. We thus look for Q-closed operators made out of ϕ,ψ, and χ. Since derivatives of fields are Q-exact we should restrict attention to local functions of the fields, and not their derivatives. It turns out we do not need the ψ fields to complete the list. The most general local function of ϕ and χ is of the form

$$\mathcal{O}_\eta = \sum_{\{I\}} \eta_{I_1,\ldots,I_p}(\phi)\chi^{I_1}\chi^{I_2}\cdots\chi^{I_p} \ , \tag{3.15}$$

with symmetrization on the indices enforced by the odd statistics of χ. As the notation suggests we have a well- defined object if

$$\eta = \sum_{\{I\}} \eta_{I_1,\ldots,I_p}(\phi) d\phi^{I_1} \wedge d\phi^{I_2} \cdots \wedge d\phi^{I_p} \tag{3.16}$$

is a p-form on M.

It is easy to compute the action of Q on $\mathcal{O}_\eta$ from (3.8). We find

$$\{Q, \mathcal{O}_\eta\} = -\mathcal{O}_{d\eta} \ , \tag{3.17}$$

so that Q acts as the exterior derivative on forms. This makes it clear that the space of observables of the **A** model, the cohomology of Q, is isomorphic as a vector space to $H^*_{DR}(M)$. There are other observables one can construct if one relaxes the condition of locality. Such integrated observables are important later when one wants to consider deformations of the model.

We note first that the requirement that an operator be Q- closed is precisely equivalent to the requirement that it be an antichiral primary operator under the holomorphic $N=2$ algebra and a chiral primary operator under the antiholomorphic algebra. Thus we have verified the claim I (should have) made last time (but was too tired and got wrong) that the (a,c) ring of the superconformal model on M is isomorphic as a vector space to $H^*(M)$.