Problem Set 4 with Solutions
Due: 3/19/08
PHY 292 Spring 2008

1 Local Conservation of Charge

Show that Maxwell's equation

$$\nabla^\mu F_{\mu\nu} = -J_\nu \tag{1.1}$$

implies the local conservation of electric charge

$$\nabla_\mu J^\mu = 0. \tag{1.2}$$

When we contract the free index in Eq.(1.1) with another covariant derivative, we find

$$\nabla_\mu J^\mu = \nabla_\mu \nabla_\nu F^{\mu\nu} = \frac{1}{2}[\nabla_\mu, \nabla_\nu]F^{\mu\nu} = \frac{1}{2}(R^\mu{}_{\lambda\mu\nu}F^{\lambda\nu} + R^\nu{}_{\lambda\mu\nu}F^{\mu\lambda}) = R_{\mu\nu}F^{\mu\nu} \tag{1.3}$$

where we have used the antisymmetry of $F_{\mu\nu}$ twice. Now observe that the Ricci tensor is symmetric, and thus the r.h.s. vanishes. This is a general feature of antisymmetric tensors and torsion-free connections: if $M^{\mu\nu}$ is an arbitrary antisymmetric tensor, then following the same steps as above we can easily show that

$$\nabla_\mu \nabla_\nu M^{\mu\nu} = 0. \tag{1.4}$$

2 Killing Vector

Show that a Killing vector satisfies the equations stated in class

$$\begin{aligned} \nabla_\mu \nabla_\nu K^\lambda &= R^\lambda{}_{\nu\mu\rho} K^\rho \\ K^\mu \nabla_\mu R &= 0. \end{aligned} \tag{2.1}$$

A Killing vector satisfies

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0. \tag{2.2}$$

To show Eqs. (2.1) we use symmetry properties of the Riemann tensor and the Bianchi identity (see Section 3.7 in Carroll)

$$R_{\rho[\sigma\mu\nu]} = 0. \tag{2.3}$$

Lowering the upper index in the first equation, we find

$$\begin{aligned} R_{\lambda\nu\mu\rho}K^\rho &= R_{\lambda\rho\mu\nu}K^\rho + R_{\lambda\mu\nu\rho}K^\rho = R_{\lambda\rho\mu\nu}K^\rho + R_{\nu\rho\lambda\mu}K^\rho \\ &= \nabla_\mu \nabla_\nu K_\lambda - \nabla_\nu \nabla_\mu K_\lambda + \nabla_\lambda \nabla_\mu K_\nu - \nabla_\mu \nabla_\lambda K_\nu \\ &= \nabla_\mu \nabla_\nu K_\lambda + \nabla_\nu \nabla_\lambda K_\mu - \nabla_\lambda \nabla_\nu K_\mu + \nabla_\mu \nabla_\nu K_\lambda \\ &= 2\nabla_\mu \nabla_\nu K_\lambda + R_{\mu\rho\nu\lambda}K^\rho \\ &= 2\nabla_\mu \nabla_\nu K_\lambda - R_{\lambda\nu\mu\rho}K^\rho. \end{aligned} \tag{2.4}$$

Now we raise the λ-index (we have a metric-compatible connection!) and bring the Riemann tensor to the r.h.s. to complete the proof.

To show the second equation in Eq.(2.1), contract the μ and λ indices in the first equation and contract the resulting equation with ∇^ν to obtain

$$\nabla_\mu \nabla_\nu K^\mu = R_{\nu\mu} K^\mu \quad \rightarrow \quad \nabla^\nu \nabla_\mu \nabla_\nu K^\mu = K^\mu \nabla^\nu R_{\nu\mu} = \frac{1}{2} K^\mu \nabla_\mu R. \tag{2.5}$$

Here we used the symmetry of $R_{\mu\nu} = R_{\nu\mu}$ to set $R_{\nu\mu} \nabla^\nu K^\mu = 0$, and we used

$$\nabla^\nu R_{\nu\mu} = \frac{1}{2} \nabla_\mu R \tag{2.6}$$

which follows from contracting the Bianchi identity $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$ (see Carroll Eq.(3.149)). We now use Eq.(2.6) in the following way:

$$\begin{aligned} K^\sigma \nabla_\sigma R = 2K^\sigma \nabla^\mu R_{\mu\sigma} = 2\nabla^\mu (K^\sigma R_{\mu\sigma}) = 2\nabla^\mu \nabla_\sigma \nabla_\mu K^\sigma = [\nabla^\mu, \nabla_\sigma] \nabla_\mu K^\sigma \\ = R^\sigma{}_{\rho\mu\sigma} \nabla^\mu K^\rho + R^\mu{}_{\rho\mu\sigma} \nabla^\rho K^\sigma = -R_{\rho\mu} \nabla^\mu K^\rho + R_{\rho\sigma} \nabla^\sigma K^\rho = 0 \end{aligned} \tag{2.7}$$

and obtain the desired result.

3 Symmetries of the Metric & Killing Vectors

(a) Show that if the metric coefficients in a given coordinate system are independent of one of the coordinates

$$\partial_\alpha g_{\mu\nu} = 0 \tag{3.1}$$

then the coordinate derivative

$$K = \partial_\alpha \tag{3.2}$$

satisfies Killing's equation.

(b) Use this result to find six Killing vectors of the flat metric on $\mathbf{R}^3$. Note that no single coordinate system can possibly realize all six of these as coordinate derivatives, so you should consider various coordinate systems with which you are familiar. Check that all the vector fields you find are in fact smooth.

(c) Compute the Lie brackets of the vector fields you found.

(a) In this case we can write K as

$$K = K^\mu \partial_\mu = \delta^\mu_\alpha \partial_\mu = \partial_\alpha \quad \Longrightarrow K^\mu = \delta^\mu_\alpha, \quad K_\mu = g_{\mu\alpha}, \quad \alpha \text{ fixed.} \tag{3.3}$$

Then the Lie derivative of the metric in the direction of K is given by

$$(\mathcal{L}_K g)_{\mu\nu} = K^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu K^\rho + g_{\nu\rho} \partial_\mu K^\rho = \partial_\alpha g_{\mu\nu} = 0, \tag{3.4}$$

showing that K is a Killing vector. Alternatively, we can compute

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = \nabla_\mu g_{\nu\alpha} + \nabla_\nu g_{\mu\alpha}. \tag{3.5}$$

Note that the index α is fixed, and thus $g_{\nu\alpha}$ transforms as a co-vector:

$$\nabla_\mu g_{\nu\alpha} = \partial_\mu g_{\nu\alpha} - \Gamma^\rho{}_{\mu\nu} g_{\rho\alpha} \ (\neq 0 \ !) \tag{3.6}$$

Then the r.h.s. of the Killing equation becomes

$$\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - 2g_{\rho\alpha} \Gamma^\rho{}_{\mu\nu} = \partial_\alpha g_{\mu\nu} = 0, \tag{3.7}$$

in agreement with the calculation using the Lie derivative.

(b) Let $g_{ij} = \delta_{ij}$ be the flat metric on $\mathbf{R}^3$. We have three Killing vectors $p_i = \partial_i$ corresponding to generators of the translation symmetries. We also have rotational symmetry, so we expect the generators of rotations, $L_i = \epsilon_{ijk} x_j p_k$, to be Killing vectors as well (note that apart from a missing $\sqrt{-1}$, we recognize $\vec{p}$ and $\vec{L}$ as our friends linear and angular momentum from Quantum Mechanics). It is straightforward to explicitly check that the L_i correspond to our notion of Killing vectors in spherical polar coordinates (r, θ, ϕ), where the metric defined by

$$ds^2 = dr^2 + r^2(d\theta^2 + sin^2 d\phi^2) \tag{3.8}$$

is independent of ϕ and ∂_ϕ is a Killing vector. Relating back to a Cartesian coordinate system we find we find the corresponding L_i (we have three by symmetry).

(c) We know from Quantum Mechanics or from brute force computation that

$$[p_i, p_j] = 0, \ [L_i, p_j] = -\epsilon_{ijk} p_k, \ [L_i, L_j] = -\epsilon_{ijk} L_k, \tag{3.9}$$

providing us with the desired Lie brackets.

4 Embedded Submanifolds: Pullbacks & Pushforwards

One application of pullbacks is the induced metric I mentioned in class. One way to construct a submanifold $N \subset M$ is to start with two abstract manifolds M, N of dimensions m, n and provide a smooth map

$$\Phi : N \to M \tag{4.1}$$

such that Φ is injective (one-to-one) and has a smooth inverse. This means the image of Φ can be thought of as a copy of N sitting inside M, and we say N is an embedded submanifold of M with Φ the embedding. This requires $m \geq n$.

Tangent vectors to N are defined by their action on functions $f : N \to \mathbf{R}$. The embedding allows us to consider the tangent space to N at a point p as a subspace of the tangent space to M at the point $\Phi(p)$, because a function $g : M \to \mathbf{R}$ defines a function $g \circ \Phi : N \to \mathbf{R}$.

(a) Show that this induces an identification of T_pN as a vector subspace of $T_{\Phi(p)}M$.

(b) Show that this identification is the same as the *pushforward* by Φ as defined in class. Choosing coordinates x^a on N and y^μ on M (locally) so that the map Φ is written as $y^\mu(x)$, find the components of $\Phi_*(v)$ in terms of the components of v for $v \in T_pN$.

(c) Using this identification, we can use a metric g on M to induce a metric γ on N, simply by defining, for any two vectors $u, v \in T_pN$

$$\gamma(u, v) = g(\Phi_* u, \Phi_* v). \tag{4.2}$$

Write this in coordinates and show that this is what we defined as the pullback of g by Φ.

(a) Let $p \in N$ be an arbitrary point. As described in the text, a tangent vector $v_p \in T_pN$ determines a tangent vector $\Phi_* v \in T_{\Phi(p)}M$ by its action on functions. For any $f : M \to \mathbf{R}$ we have

$$\Phi_* v(f) = v(\Phi^* f), \tag{4.3}$$

where $\Phi^* f : N \to \mathbf{R}$ is just

$$\Phi^* f = f \circ \Phi. \tag{4.4}$$

To show that this is indeed a tangent vector, we need to show that the action on functions is linear

$$\Phi_* v(f+g) = v(\Phi^*(f+g)) = v(\Phi^* f + \Phi^* g)) = v(\Phi^* f) + v(\Phi^* g) = \Phi_* v(f) + \Phi_* v(g), \tag{4.5}$$

where the second equality used the fact that

$$\Phi^*(f+g) = (f+g) \circ \Phi = f \circ \Phi + g \circ \Phi. \tag{4.6}$$

We also need to show that it satisfies the Leibniz rule

$$\begin{aligned} \Phi_* v(fg) &= v(\Phi^*(fg)) = v\left((\Phi^* f)(\Phi^* g)\right) \\ &= \Phi^* f(p) v(\Phi^* g) + \Phi^* g(p) v(\Phi^* f) = f(\Phi(p)) \Phi_* v(g) + g(\Phi(p)) \Phi_* v(f). \end{aligned} \tag{4.7}$$

The set of all vectors in $T_{\Phi(p)}M$ obtained in this way will be a vector subspace if we show that the sum of two members of the subset is itself contained in the subset. This is also clear, since for $v, w \in T_pN$

$$\Phi_* v + \Phi_* w = \Phi_*(v + w) \ . \tag{4.8}$$

The fact that this subspace is isomorphic to T_pN follows from the fact that Φ_* is injective, which in turn follows from the injectivity of Φ.

(b) Explicitly in coordinates x^α on N and y^μ on M we have, for $v = v^\alpha \frac{\partial}{\partial x^\alpha}$

$$\Phi_* v = \frac{\partial \Phi^\mu}{\partial x^\alpha} v^\alpha \frac{\partial}{\partial y^\mu}. \tag{4.9}$$

(c) Using our expression above we have

$$\gamma(u, v) = \gamma_{\alpha\beta} u^\alpha v^\beta, \tag{4.10}$$

with the components determined by

$$\gamma(u, v) = g(\Phi_* u, \Phi_* v) = g_{\mu\nu} \frac{\partial \Phi^\mu}{\partial x^\alpha} u^\alpha \frac{\partial \Phi^\nu}{\partial x^\beta} v^\beta. \tag{4.11}$$

Thus

$$\gamma_{\alpha\beta} = g_{\mu\nu} \frac{\partial \Phi^\mu}{\partial x^\alpha} \frac{\partial \Phi^\nu}{\partial x^\beta}, \tag{4.12}$$

is the pullback $\Phi^*(g)$ of g.

5 The Torus

In cylindrical coordinates (ρ, θ, z), the metric on flat Euclidean 3-space takes the form $ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2$. The surface defined by $(\rho - a)^2 + z^2 = b^2$ for $a > b > 0$ describes the embedding of a 2-torus in this space.

(a) Using the coordinates ρ, θ to parametrize the torus, determine the metric γ_{ij} on the torus induced by this embedding. Use this to calculate the surface area $\int \sqrt{|\gamma|}$ of the torus (where $|\gamma|$ is the determinant of the metric).

(b) Compute the Riemann tensor, Ricci tensor, and Ricci scalar for this metric, and use this to calculate $\int \sqrt{|\gamma|} R$ for this metric.

(c) Ignoring its above realization as an embedded surface in $\mathbf{R}^3$, show that it is also possible to put a *flat* metric on the 2-torus (note that this is a distinct metric, not a change of coordinates, because the curvature is different). Use the fact that the Euler character $\chi = \int \sqrt{|g|} R$ is a topological invariant, independent of the choice of metric, to check your result for (a).

(a) We write the embedding map here as

$$\Phi : (\rho, \theta) \mapsto (\rho, z, \theta) \tag{5.1}$$

where

$$z^2 = b^2 - (\rho - a)^2 \tag{5.2}$$

to find on the torus

$$\frac{\partial z}{\partial \rho} = -\frac{\rho - a}{\sqrt{b^2 - (\rho - a)^2}} \tag{5.3}$$

so that we have

$$\gamma_{\rho\rho} = g_{\rho\rho} + g_{zz} \left(\frac{\partial z}{\partial \rho} \right)^2 = 1 + \frac{(\rho - a)^2}{b^2 - (\rho - a)^2} = \frac{b^2}{b^2 - (\rho - a)^2}. \tag{5.4}$$

We also have clearly $\gamma_{\theta\theta} = \rho^2$ and all other components vanish. The determinant is thus

$$|\gamma| = \frac{\rho^2 b^2}{b^2 - (\rho - a)^2} \tag{5.5}$$

and integrating over the region $|\rho - a| \leq b$ we find

$$A = 4\pi b \int_{a-b}^{a+b} \frac{\rho d\rho}{\sqrt{b^2 - (\rho - a)^2}} = 4\pi^2 ab. \tag{5.6}$$

(b) Let's start with the nonzero connection coefficients

$$\Gamma^{\rho}_{\rho\rho} = \frac{a - \rho}{b^2 - (\rho - a)^2}, \quad \Gamma^{\theta}_{\rho\theta} = \frac{1}{\rho}, \quad \Gamma^{\rho}_{\theta\theta} = -\frac{\rho(b^2 - (\rho - a)^2)}{b^2}. \tag{5.7}$$

Now we can compute the one nontrivial Riemann tensor component

$$R^{\rho}_{\theta\rho\theta} = \partial_\rho \Gamma^{\rho}_{\theta\theta} - \partial_\theta \Gamma^{\rho}_{\theta\rho} + \Gamma^{\rho}_{\alpha\rho}\Gamma^{\alpha}_{\theta\theta} - \Gamma^{\rho}_{\alpha\theta}\Gamma^{\alpha}_{\theta\rho} = \partial_\rho \Gamma^{\rho}_{\theta\theta} + \Gamma^{\rho}_{\rho\rho}\Gamma^{\rho}_{\theta\theta} - \Gamma^{\rho}_{\theta\theta}\Gamma^{\theta}_{\theta\rho} = \frac{\rho(\rho - a)}{b^2}. \tag{5.8}$$

Contracting, we find

$$R_{\theta\theta} = \frac{\rho(\rho - a)}{b^2}, \quad R_{\rho\rho} = \frac{\rho - a}{\rho(b^2 - (\rho - a)^2)}. \tag{5.9}$$

(b) *(ctd.)* Contracting one last time, we have

$$R = \frac{2(\rho - a)}{b^2 \rho}. \tag{5.10}$$

We can now integrate to find

$$\int \sqrt{|\gamma|} R = 4\pi \int_{a-b}^{a+b} \frac{(\rho - a) d\rho}{b\sqrt{b^2 - (\rho - a)^2}} = 0, \tag{5.11}$$

since the integrand is clearly odd under $\rho \to 2a - \rho$.

(c) A torus is parametrized by two angles. We can take them in the coordinates given above as θ and $\psi = \tan^{-1}(z/(\rho - a))$. In terms of these we can simply choose

$$ds^2 = R_1^2 d\theta^2 + R_2^2 d\psi^2 \tag{5.12}$$

for any radii $R_{1,2}$ we like. This is clearly a smooth metric on the torus, and it is also clearly flat, so that all curvature components vanish. Since we *can* choose a flat metric on the torus, and since χ is a topological invariant independent of our choice of metric, it is a good thing we found above that it vanishes!

6 The Sphere

Consider the round metric on the 2-sphere $ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)$. Find the components of the Christoffel connection in (θ, ϕ) coordinate basis, the Riemann tensor, Ricci tensor, and Ricci scalar. Write the geodesic equation and use it to find the circumference of the 2-sphere.

Let $x^1 = \theta$ and $x^2 = \phi$. The nonzero Christoffel symbols for the Levi-Civita connection are given by

$$\begin{aligned} \Gamma^1{}_{ij} &= \frac{1}{2a^2}(\partial_i g_{j1} + \partial_j g_{i1} - \partial_1 g_{ij}) \Longrightarrow \\ \Gamma^1{}_{22} &= -\frac{1}{2}\partial_\theta sin^2\theta = -\sin\theta\cos\theta \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} \Gamma^2{}_{ij} &= \frac{1}{2a^2 sin^2\theta}(\partial_i g_{j2} + \partial_j g_{i2} - \partial_2 g_{ij}) \Longrightarrow \\ \Gamma^1{}_{12} = \Gamma^1{}_{21} &= \frac{1}{2\sin^\theta}\partial_\theta sin^2\theta = \cot\theta. \end{aligned} \tag{6.2}$$

The Riemann tensor is easily computed to be

$$R^1{}_{212} = \sin^2\theta, \quad R_{1212} = a^2\sin^2\theta. \tag{6.3}$$

The non-zero components of the Ricci tensor are

$$R_{11} = 1, \quad R_{22} = sin^2\theta, \tag{6.4}$$

and the Ricci scalar is given by

$$R = \frac{2}{a^2}. \tag{6.5}$$

The geodesic equations for θ and ϕ are

$$\begin{aligned}\ddot{\theta} - \sin\theta\cos\theta\ \dot{\phi}^2 = 0 \\ \ddot{\phi} + 2\cot\theta\ \dot{\theta}\dot{\phi} = 0\end{aligned} \tag{6.6}$$

Let $\gamma(t) = (\theta(t), \phi(t)$ be a geodesic. Choose initial conditions $\gamma(0) = (\frac{\pi}{2}, 0)$ and $\dot{\gamma}(t) = (0, \omega)$ for some constant $\omega > 0$. Then $\gamma(t) = (\frac{\pi}{2}, \omega t)$ is obviously a solution to Eqs.(6.6) and intersects itself at time $T = \frac{2\pi}{\omega}$. We have

$$|\dot{\gamma}(t)|^2 = g_1 1\dot{\theta}^2 + g_2 2\dot{\phi}^2 = a^2\omega^2, \tag{6.7}$$

and thus we can compute the circumference of S^2 as the length of γ:

$$L(\gamma) = \int_0^T |\dot{\gamma}| dt = \int_0^{\frac{2\pi}{\omega}} a\omega dt = 2\pi a. \tag{6.8}$$