Physics 292

Final Exam

Due 5/1/08

The exam is due by noon Thursday. You can hand it in by sliding it under my office door (if the office is locked) or simply putting it on my chair (if not). You can also email me your solutions to plesser@cgtp.duke.edu . Drop-off at my home (somewhat closer to Chapel Hill for those coming from there and not requiring Dukecard access) can be arranged by contacting me.

You may use the two texts for the class (Carroll and Wald) as well as your class notes.

I have tried hard to keep the exam free of errors but you know me well enough by now to realize this is not my forte. If you think you have found an error, or need to contact me for any reason, you can reach me by email (I will try to check as often as I can) or by phone at (919) 414-7669. Please try not to call after 11pm or before 8am.

1. Gravitational lensing: In this problem we will consider the deflection of light by a massive object in a FRW universe. This is relevant when we study the lensing of cosmologically distant objects such as quasars. We consider a situation in which a massive galaxy is located directly between the observer (us) and the distant quasar. Because of the deflection of light by the galaxy, we observe the light from the quasar as a ring around the galaxy whose diameter subtends an angle 2θ. We will model the process by pretending that light moves along a null geodesic of the background FRW metric until it is "instantaneously" deflected by the galaxy, then proceeds along a new null geodesic to us. If the distance of closest approach between the light beam and the galaxy is large compared to the size of the galaxy - so we can model the galaxy as a point mass M - but much smaller than the Hubble scale, we can treat the deflection as occuring in a Schwarzschild geometry.

   (a) Suppose the FRW background is matter-dominated $ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\Omega)$ with $a(t) = a_0(t/t_0)^{2/3}$. The quasar emits the radiation at time t_2; it is then deflected at time t_1 and reaches us now (t_0). Find an expression for θ in terms of M, t_0, t_1, and t_2.

(b) Re-express your result in terms of physically observed parameters: M, the redshifts z_1 and z_2 of the galaxy and the quasar, and the Hubble time H_0.

2. In the presence of and electromagnetic field, a particle of charge e and mass m moves along a trajectory determined by

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = \frac{e}{m}F^\mu{}_\nu\frac{dx^\nu}{d\tau} .$$

Consider such a particle moving in the field of a Reissner-Nordström black hole of charge Q and mass M.

(a) Show that the energy

$$E = m\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)\frac{dt}{d\tau} + \frac{eQ}{r}$$

is conserved along the trajectory.

(b) Use this to describe a Penrose-like process for a charged black hole. Find an inequality relating δM to δQ for the process you describe.

(c) Show that this is compatible with Hawking's area theorem.

(d) Starting with an extremal black hole, what fraction of its mass can be extracted using this process?

3. Certain field theories predict the existence of relstivistically invariant "defects" which behave rather differently than ordinary massive objects. By symmetry arguments, the energy-momentum tensor for a massive domain wall (two-dimensional defect) perpendicular to the z-axis may be written in its rest frame as $T^{\mu\nu} = \delta(z)\mathrm{diag}(\sigma, p, p, 0)$, with σ the surface energy density and p the pressure ($-p$ is the surface tension).

(a) Show that this becomes invariant under Lorentz boosts in the (x, y) plane for $p = -\sigma$.

(b) Show that in the weak-field approximation, the Einstein equations are solved by

$$h_{00} = -4\pi G(\sigma + 2p)|z|$$
$$h_{33} = -4\pi G(\sigma - 2p)|z|$$
$$h_{11} = h_{22} = -4\pi G\sigma|z| ,$$

all other components vanishing. Where does the weak-field approximation fail?

(c) Show that the geodesic equation of motion for a slow (nonrelativistic) test particle in a weak gravitational field reduces to

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{00} - 2\Gamma^i_{0j}\frac{dx^j}{dt} \ ,$$

and use this to compute the acceleration of a test particle in the vicinity of the domain wall. Compare the results for a wall of pressureless dust ($p = 0$) to the relativistic case ($p = -\sigma$).

(d) In which direction is light deflected for each of these extreme cases?

(e) For the relativistic case, show that a gauge transformation

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

for

$$\xi_3 = f(z)$$

and other components vanishing, brings the metric to a conformally flat form, by finding the appropriate function $f(z)$. How does this help explain your result for (d)?