Physics 342

Problem set 1 Solutions

1. We start with our free Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\left(\partial_\mu \phi\right)^2 - \tfrac{1}{2}\mu^2\phi^2 \ ,$$

and the field A defined by

$$\phi = A + \tfrac{1}{2}gA^2 \ .$$

In terms of A we have

$$\mathcal{L} = \tfrac{1}{2}\left(\partial_\mu A\right)^2 (1+gA)^2 - \tfrac{1}{2}\mu^2\left(A + \tfrac{1}{2}gA^2\right)^2 \ .$$

Expanding this we write

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}' \ ,$$

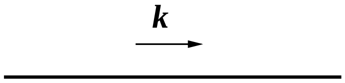
where

$$\mathcal{L}_0 = \tfrac{1}{2}\left(\partial_\mu A\right)^2 - \tfrac{1}{2}\mu^2 A^2$$

is the free Lagrangian and

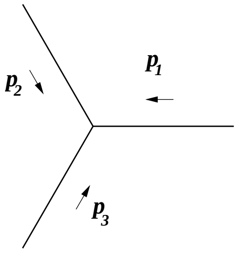
$$\mathcal{L}' = g\left(A(\partial_\mu A)^2 - \tfrac{1}{2}\mu^2 A^3\right) + \tfrac{1}{2}g^2\left(A^2(\partial_\mu A)^2 - \tfrac{1}{4}\mu^2 A^4\right)$$

is to be considered an interaction. Our first task is to write the Feynman rules which follow from this Lagrangian. From $\mathcal{L}_0$ we extract the usual propagator



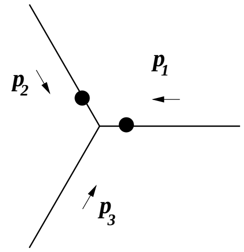
$$\frac{i}{k^2 - \mu^2 + i\epsilon}$$

Now for the interactions. Let's tackle the cubic interactions at order g first. There are two terms in $\mathcal{L}'$ and correspondingly two types of vertices. Let's start with the second term. This is of a type we have seen before and leads directly to

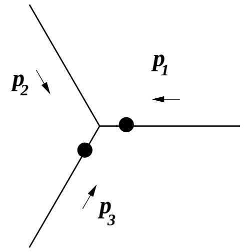


$$-ig\frac{\mu^2}{2}3!(2\pi)^4\delta^{(4)}(p_1+p_2+p_3)$$

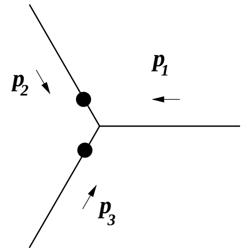
where the only factor possibly requiring explanation is 3!, which counts the six different contractions that arise when we insert this term into a diagram, corresponding to the different ways to contract the three field operators (legs) to other operators in the graph. The other cubic term in $\mathcal{L}'$ contains derivative interactions. Taking the hint and assuming all I need to remember is that each incoming momentum gives a factor of $-ip_\mu$ to the amplitude, we find that the six contraction split into three sets of two, depending where the derivatives act. I denote these legs in the figure by dots on the lines. We get



$$ig\,2!\,(-ip_1)(-ip_2)(2\pi)^4\delta^{(4)}(p_1+p_2+p_3)$$

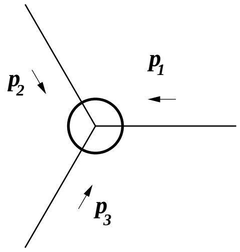


$$ig\,2!\,(-ip_1)(-ip_3)(2\pi)^4\delta^{(4)}(p_1+p_2+p_3)$$



$$ig\,2!\,(-ip_2)(-ip_3)(2\pi)^4\delta^{(4)}(p_1+p_2+p_3)$$

When inserting all of these in a larger graph they are completely interchangable, since they have the same number of legs and impose the same constraint on the momenta. So it is useful to sum them into one “effective” cubic vertex, incorporating all of them

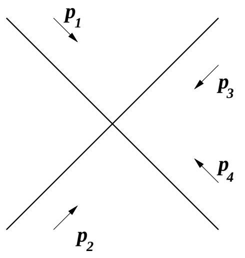


$$\begin{aligned}&-ig(3\mu^2+2(p_1\cdot p_2+p_1\cdot p_3+p_2\cdot p_3)(2\pi)^4\delta^{(4)}(p_1+p_2+p_3)\\&=-ig(3\mu^2-p_1^2-p_2^2-p_3^2)(2\pi)^4\delta^{(4)}(p_1+p_2+p_3)\end{aligned}$$

where to simplify the expression I have used the fact that under the δ function we have

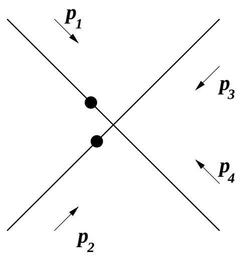
$$0 = (p_1 + p_2 + p_3)^2 = p_1^2 + p_2^2 + p_3^2 + 2(p_1 \cdot p_2 + p_1 \cdot p_3 + p_2 \cdot p_3) \ .$$

Now we can treat the quartic interaction at order g^2 the same way. There are once more two quartic terms in $\mathcal{L}'$ and so once more we have two types of vertices. The first, coming from the second term, is simply



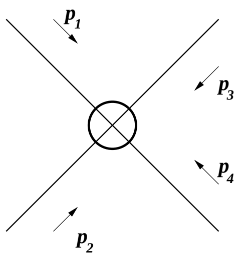
$$-ig^2 \frac{\mu^2}{8}\, 4!\, (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 + p_4) \ .$$

When evaluating the other quartic term, the 24 contractions split into six sets of four, with the members of each set related by exchanging legs with and without dots among themselves. Thus we get, for example,



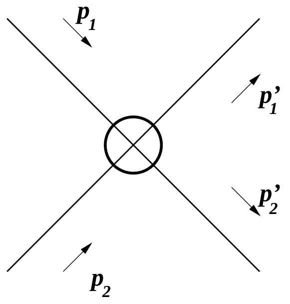
$$i\frac{g^2}{2}\, (2!)^2\, (-ip_1)(-ip_2)(2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 + p_4) \ ,$$

and five more such diagrams. As before we can sum all of these to an "effective" quartic vertex



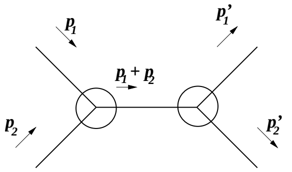
$$\begin{aligned} &- ig^2 \big(3\mu^2 + 2 \sum_{i<j} p_i \cdot p_j\big)(2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 + p_4) \\ &= -ig^2 \big(3\mu^2 - \sum_i p_i^2\big)(2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 + p_4) \ , \end{aligned}$$

where I have again used momentum conservation to simplify. Now that we have the Feynman rules under control, we can write the diagrams that contribute to elastic scattering to order $\mathcal{O}(g^2)$. As usual, we denote the incoming momenta by $p_{1,2}$ and the outgoing momenta by $p'_{1,2}$. All of these have positive time component, which tells us the outgoing momenta need to be inverted relative to the conventions in my vertices. All of these, being on-shell, square to μ^2; as mentioned in the problem set, this will play a role in the cancellations we need and I will use it freely. We will have one (in terms of our effective vertices) diagram with one quartic vertex

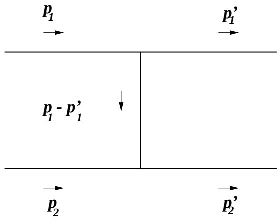


$$i\mathcal{A} = -ig^2(3\mu^2 - 4\mu^2) = ig^2\mu^2 \ .$$

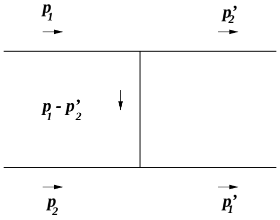
There are also three diagrams at this order made from two cubic vertices. Using our effective vertices these are



$$\begin{aligned} i\mathcal{A} &= (-ig)(3\mu^2 - 2\mu^2 - s)\frac{i}{s-\mu^2}(-ig)(3\mu^2 - 2\mu^2 - s) \\ &= -ig^2(s - \mu^2) \ . \end{aligned}$$



$$\begin{aligned} i\mathcal{A} &= (-ig)(3\mu^2 - 2\mu^2 - t)\frac{i}{t-\mu^2}(-ig)(3\mu^2 - 2\mu^2 - t) \\ &= -ig^2(t - \mu^2) \ . \end{aligned}$$



$$\begin{aligned} i\mathcal{A} &= (-ig)(3\mu^2 - 2\mu^2 - u)\frac{i}{u-\mu^2}(-ig)(3\mu^2 - 2\mu^2 - u) \\ &= -ig^2(u - \mu^2) \ . \end{aligned}$$

Adding all four contributions we have then

$$i\mathcal{A} = -ig^2(s + t + u - 4\mu^2) = 0 \ .$$

2. Let me start this one with a review of density-of-states computations, since I did not do this in class and some people seemed to have some difficulty with it. We begin with the statement that the transition probability is simply the square of the amplitude

$$P_{fi} = |\langle f|(S-1)|i\rangle|^2 \ .$$

In our notation,

$$\langle f|(S-1)|i\rangle = i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(P_f - P_i) \ ,$$

Considering an experiment taking place in a world of volume V over time T, we use

$$(2\pi)^4\delta^{(4)}_{VT}(P_f - P_i) = \int_{VT} d^4x\, e^{i(P_f-P_i)\cdot x} \ ,$$

so that

$$\left|(2\pi)^4\delta^{(4)}_{VT}(P)\right|^2 = VT(2\pi)^4\,\delta^{(4)}_{VT}(P) \ ,$$

as can be seen by computing

$$\int\frac{d^4p}{(2\pi)^4}\left|(2\pi)^4\delta^{(4)}_{VT}(P)\right|^2 = \int_{VT} d^4x = VT \ .$$

Thus we have

$$P_{fi} = VT|\mathcal{A}_{fi}|^2(2\pi)^4\,\delta^{(4)}(P_f - P_i) \ .$$

The linear dependence on time is simple to understand. If our states are time-independent then the process proceeds at a constant *rate*. The volume factor needs to be considered more carefully. These computations used relativistically normalized states, i.e. such that one-particle states satisfy

$$\langle p|p'\rangle = 2p^0(2\pi)^3\delta^{(3)}(\mathbf{p}' - \mathbf{p}) \ .$$

With this normalization, the state $|p\rangle$ in fact corresponds to a particle density of $2p^0$ particles per unit volume. Since we are trying to compute the probability of finding one particle for each momentum, we should use

$$\langle f|(S-1)|i\rangle = i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(P_f - P_i)\prod_{\text{out}}(2E_iV)^{-1/2}\prod_{\text{in}}(2E_jV)^{-1/2} \ .$$

This leads to

$$P_{fi} = VT|\mathcal{A}_{fi}|^2(2\pi)^4\,\delta^{(4)}(P_f - P_i)\prod_{\text{out}}(2E_iV)^{-1}\prod_{\text{in}}(2E_jV)^{-1} \ .$$

Thus, we write

$$\frac{dP_{fi}}{d^3k_1\cdots d^3k_n\,dt} = V|\mathcal{A}|^2\prod_{\text{in}}(2E_jV)^{-1}D \ ,$$

where the density of states is give by

$$D = (2\pi)^4\, \delta^{(4)}(P_f - P_i) \prod_{\text{out}} \frac{d^3 k_i}{(2\pi)^2 (2E_i)} \ .$$

In this neat trick the volume factors disappear into the d^3k factors; the density of momentum states at finite V is proportional to V. In the limit of large V the ratio tends to a constant which is what we call d^3k. When computing a decay rate we have a single-particle incoming state. Thus the factors of V cancel immediately and we have for the rate

$$\Gamma = \frac{1}{2E} \sum_f \int |\mathcal{A}_{fi}|^2 D_f \ ,$$

where the sum is over all decay channels and D_f is the corresponding density of states. Note that in the rest frame, we have $E = m$. Thus the rate in any other frame is reduced by m/E, expressing time-dilation as expected. For a two-particle incoming state, we define the differential cross-section $d\sigma$ as the transition probability per incoming particle per target particle (for a fixed-target experiment), or equivalently the transition rate per unit flux. Since our states are normalized to have one particle, the flux is simply given by the relative velocity $|\mathbf{v}_1 - \mathbf{v}_2|/V$. We thus have

$$d\sigma = \frac{1}{4E_1 E_2 |\mathbf{v}_1 - \mathbf{v}_2|} |\mathcal{A}_{fi}|^2 D$$

(note that the factors of V have again cancelled as they must). The denominator can be written in a slightly nicer form as

$$4E_1 E_2 |\mathbf{v}_1 - \mathbf{v}_2| = 4|E_2 \mathbf{p}_1 - E_1 \mathbf{p}_2| \ ,$$

using the fact that $\mathbf{v} = \mathbf{p}/E$. For a two-particle outgoing state we have as well (in COM frame where $P_i = P_f = (E_T, \mathbf{0})$)

$$D = (2\pi)^4 \frac{d^3\mathbf{p}_1}{(2\pi)^3 2E_1} \frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2} \delta^{(3)}(\mathbf{p}_1 + \mathbf{p}_2) \delta(E_1 + E_2 - E_T) \ .$$

Using the δ function to eliminate p_2, and writing $d^3\mathbf{p}_1 = p^2 dp\, d\Omega$ with $d\Omega = \sin\theta d\theta d\phi$ a solid angle element, we have

$$D = \frac{1}{16\pi^2 E_1 E_2} p^2 dp\, d\Omega\, \delta(E_1 + E_2 - E_T) \ .$$

We now want to fix p using the final δ function, so write

$$p^2 dp\, \delta(\sqrt{p^2+m_1^2} + \sqrt{p^2+m_2^2} - E_T) = (p/E_1 + p/E_2)\, \delta(p - p_f)$$

where p_f is the value of p solving energy conservation and I have used

$$\frac{dE}{dp} = \frac{p}{E}$$

for $E = \sqrt{p^2+m^2}$. Plugging this in we have finally

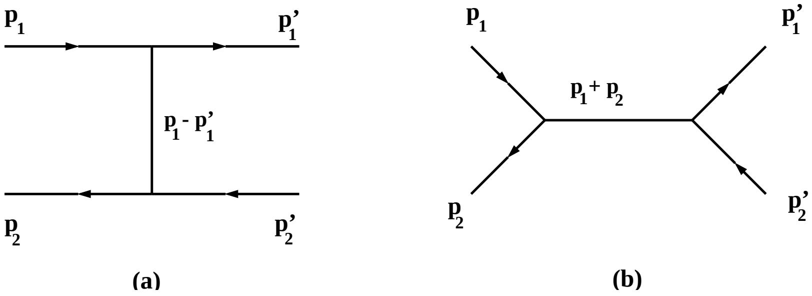
$$D = \frac{p_f d\Omega}{16\pi^2 E_T} \ .$$

In the COM frame we have $4|E_2\mathbf{p}_1 - E_1\mathbf{p}_2| = 4E_T p_i$, so

$$d\sigma = \frac{d\Omega}{64\pi^2 E_T^2} \frac{p_f}{p_i} |\mathcal{A}_{fi}|^2 \ .$$

Now to complete the computation we need only the matrix elements.

(a) For nucleon-antinucleon elastic scattering the lowest-order diagrams are



Nucleon-antinucleon elastic scattering

The matrix element is thus

$$i\mathcal{A} = (-ig)^2 \left(\frac{i}{(p_1 - p_1')^2 - \mu^2} + \frac{i}{(p_1 + p_2)^2 - \mu^2} \right) \ ,$$

where assuming $\mu < 2m$ I have dropped the $i\epsilon$ terms since the denominators cannot vanish. In the COM frame,

$$p_1 = (\sqrt{p^2+m^2}, p\mathbf{e}) \qquad p_1' = (\sqrt{p^2+m^2}, p\mathbf{e}')$$
$$p_2 = (\sqrt{p^2+m^2}, -p\mathbf{e}) \qquad p_2' = (\sqrt{p^2+m^2}, -p\mathbf{e}') \ ,$$

where $p_f = p_i = p$ and $E_T^2 = 4(p^2+m^2)$ is the total energy. Thus

$$(p_1 - p_1')^2 = -2p^2(1-\cos\theta) \qquad (p_1+p_2)^2 = E_T^2 \ .$$

Plugging this in we have

$$\frac{d\sigma}{d\Omega} = \frac{g^4}{256\pi^2(p^2+m^2)}\left|\frac{-1}{2p^2(1-\cos\theta)+\mu^2} + \frac{1}{4(p^2+m^2)-\mu^2}\right|^2 \ .$$

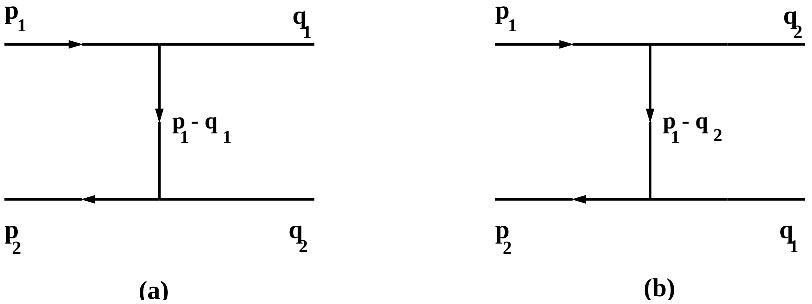
To find the total cross-section σ we need to integrate

$$\sigma = \int \sin\theta\, d\theta\, d\phi \frac{d\sigma}{d\Omega} = 2\pi\int_{-1}^{1} d\cos\theta\,\frac{d\sigma}{d\Omega} \ .$$

Expanding the square and performing the integrals we have

$$\sigma = \frac{g^4}{64\pi(p^2+m^2)}\left(\frac{1}{4(p^2+m^2)-\mu^2)^2} + \frac{1}{2p^2(4(p^2+m^2)-\mu^2)}\ln\frac{\mu^2}{4p^2+\mu^2} + \frac{1}{4\mu^2(p^2+\mu^2)}\right) \ .$$

(b) For annihilation into two mesons the relevant diagrams are



Nucleon-antinucleon annihilation to two mesons

The matrix element is

$$i\mathcal{A} = (-ig)^2\left(\frac{i}{(p_1-q_1)^2-m^2} + \frac{i}{(p_1-q_2)^2-m^2}\right) \ .$$

In COM frame we have here

$$\begin{aligned} p_1 &= (\sqrt{p^2+m^2}, p\mathbf{e}) \qquad & q_1 &= (\sqrt{q^2+\mu^2}, q\mathbf{e}') \\ p_2 &= (\sqrt{p^2+m^2}, -p\mathbf{e}) \qquad & q_2 &= (\sqrt{q^2+\mu^2}, -q\mathbf{e}') \ , \end{aligned}$$

where $p^2+m^2 = q^2+\mu^2 = E_T^2/4$. Thus

$$(p_1-q_1)^2 = -(p^2+q^2) + 2pq\cos\theta \qquad (p_1-q_2)^2 = -(p^2+q^2) - 2pq\cos\theta \ ,$$

with $\mathbf{e}\cdot\mathbf{e}' = \cos\theta$. Plugging in, we have

$$\begin{aligned} i\mathcal{A} &= (-ig)^2 \left(\frac{i}{2pq\cos\theta - (p^2+q^2) - m^2} - \frac{i}{2pq\cos\theta + (p^2+q^2) + m^2} \right) \\ &= -ig^2 \frac{2(p^2+q^2+m^2)}{4p^2q^2\cos^2\theta - (p^2+q^2+m^2)^2} \ . \end{aligned}$$

This can be simplified a little if we note that $p^2+q^2+m^2 = E_T^2/2 - \mu^2$. Plugging in we have

$$\frac{d\sigma}{d\Omega} = \frac{g^4}{64\pi^2 E_T^2}\frac{q}{p}\left| \frac{E_T^2 - 2\mu^2}{4p^2q^2\cos^2\theta - (E_T^2/2-\mu^2)^2} \right|^2 \ .$$

To get the total cross-section we need to integrate as above. Here, however, the fact that the two particles in the final state are identical bosons tells us that $|q_1\ q_2\rangle = |q_2\ q_1\rangle$. In terms of our angular variables, this means that $\mathbf{e}'$ and $-\mathbf{e}'$ describe the same state. So integrating over all values of ϕ and θ as we did above will precisely double-count every contribution. We can either restrict the integral or simply divide everything by two to find

$$\sigma = \frac{g^4}{64\pi E_T^2}\frac{q}{p}\left(\frac{4}{(\mu^2+E_T^2/2)^2 - 4p^2q^2} + \frac{1}{pq(E_T^2-2\mu^2)} \ln \frac{E_T + 2\mu^2 + 4pq}{E_T^2 + 2\mu^2 - 4pq} \right) \ .$$

3. For a decay width we have

$$\Gamma = \frac{1}{2E}\sum_f \int |\mathcal{A}_{fi}|^2 D_f \ ,$$

where the sum is over all possible final states and D_f is the density of states for the final state f. In the rest frame (in which lifetimes are defined) $E = m_K$. In our case we are considering the partial width into charged pions, so at lowest order the fundamental vertex is the only contributing diagram and

$$i\mathcal{A} = -ig + \mathcal{O}(g^2) \ .$$

We found above the density of states for a two-particle final state

$$D_2 = \frac{p_f\, d\Omega}{16\pi^2 E_T} \ .$$

In our case, energy conservation yields $m_\pi^2 + p_f^2 = m_K^2/4$, while the integral over solid angles is trivial because the amplitude is independent of angles (as it has to be for the decay of a scalar into scalars). Thus, in all,

$$\Gamma = \frac{g^2}{32\pi^2 m_K^2}(4\pi)\sqrt{m_K^2/4 - m_\pi^2} = \frac{\sqrt{m_K^2/4 - m_\pi^2}}{8\pi}\left(\frac{g}{m_K}\right)^2 \ .$$

Plugging in the numbers given in the problem, we find

$$\frac{g}{m_K} \sim 7.9 \times 10^{-7} \ .$$