**Physics 342**

Solutions to Problem set 2

1.

(a) In class, we computed $\Pi'$ to order $\mathcal{O}(g^2)$. Of course, there was an error in the calculation, but after taking the correction into account, we have

$$\Pi'(k^2) = -\frac{g^2}{4\pi^2}\int_0^1 dx \left\{ \left(\alpha m^2 - 3x(1-x)k^2\right) \log \frac{m^2 - k^2 x(1-x) - i\epsilon}{m^2 - \mu^2 x(1-x) - i\epsilon} \right.$$
$$\left. + x(1-x)(k^2-\mu^2)\frac{\alpha m^2 - 3\mu^2 x(1-x) - i\epsilon}{m^2 - \mu^2 x(1-x) - i\epsilon} \right\} ,$$

with

$$\alpha = \begin{cases} 3 & \Gamma = 1 \\ 1 & \Gamma = i\gamma_5 \, . \end{cases}$$

Since $\mu < 2m$ and $0 \leq x(1-x) \leq 1/4$, the denominator $m^2 - \mu^2 x(1-x) - i\epsilon$ lies below the positive real axis and does not approach the origin. In the second term, the limit $\epsilon \to 0$ is thus real, and this term does not contribute to $\mathrm{Im}(\Pi')$. In the first term, we use the fact that $\mathrm{Im}(\log x) = \arg x$. As mentioned above, the denominator, which lies below the positive real axis, has argument $2\pi$ (I choose the branch cut of the logarithm to lie along the positive real axis; any other choice leads to identical answers). For $k^2 < 4m^2$, the numerator also has argument $2\pi$ for any $x$ in the interval. Thus for $k^2 < 4m^2$ the integral is real. For larger values of $k^2$, there is an interval

$$x \in [\tfrac{1}{2} - \sqrt{1/4 - m^2/k^2}, \tfrac{1}{2} + \sqrt{1/4 - m^2/k^2}]$$

for which the numerator lies below the *negative* real axis. For these values it has argument $\pi$ and the quotient thus has argument $-\pi$. Thus

$$\mathrm{Im}\left(\Pi'(k^2)\right) = -\frac{g^2}{4\pi^2}\int_{\frac{1}{2}-\sqrt{1/4-m^2/k^2}}^{\frac{1}{2}+\sqrt{1/4-m^2/k^2}} dx \, \left(\alpha m^2 - 3x(1-x)k^2\right)(-\pi)$$
$$= \frac{g^2}{4\pi}\sqrt{1/4 - m^2/k^2}\left(2(\alpha-1)m^2 - k^2\right)\theta(k^2 - 4m^2) \, .$$

(b) For $k^2 < 4m^2$, we have $\sigma(k^2) = 0$ (showed in class) as well as $\mathrm{Im}\left(\Pi'(k^2)\right) = 0$ (showed above). So let $k^2 \geq 4m^2 > \mu^2$. The spectral representation is

$$D'(k) = \frac{i}{k^2 - \mu^2 + i\epsilon} + \int da^2 \, \frac{i\sigma(a^2)}{k^2 - a^2 + i\epsilon} \; .$$

For $k^2 > \mu^2$, and using the fact that $\sigma$ is manifestly real, we have

$$2\mathrm{Re}\left(D'(k^2)\right) = i \int da^2 \, \sigma(a^2) \left( \frac{1}{k^2 - a^2 + i\epsilon} - \frac{1}{k^2 - a^2 - i\epsilon} \right) \; .$$

Using our identity from the hint,

$$\lim_{\epsilon \to 0} \left( \frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon} \right) = -2\pi i \delta(x) \; ,$$

we find

$$\mathrm{Re}\left(D'(k^2)\right) = \pi \sigma(k^2) \; .$$

On the other hand, we have

$$D'(k^2) = \frac{i}{k^2 - \mu^2 - \Pi'(k^2) + i\epsilon} \; .$$

Thus

$$\begin{aligned} 2\,\mathrm{Re}\left(D'(k^2)\right) &= \frac{i}{k^2 - \mu^2 - \Pi'(k^2) + i\epsilon} - \frac{i}{k^2 - \mu^2 - \overline{\Pi'}(k^2) - i\epsilon} \\ &= i \frac{\Pi'(k^2) - \overline{\Pi'}(k^2) - 2i\epsilon}{|k^2 - \mu^2 - \Pi'(k^2)|^2} \\ &= -2\,\mathrm{Im}\,\Pi'(k^2)\,|D'(k^2)|^2 \; . \end{aligned}$$

Putting it together we have

$$\mathrm{Im}\left(\Pi'(k^2)\right) = -\pi |D'(k^2)|^{-2} \sigma(k^2) \; .$$

(c) The spectral density is given by

$$\sigma(k^2)\theta(k^0) = \sum_{n \geq 2} (2\pi)^3 \delta^{(4)}(k - P_n) |\langle n|\phi'(0)|0\rangle|^2 \; .$$

The summation $\sum_n$ is symbolic. It represents a sum over all eigenstates of the Hamiltonian except the single-particle states. Thus, it includes in particular integrals over (*on-shell*) momenta for the $n$ particles. Using the hint from the assignment, we have, for example,

$$
\begin{aligned}
&{}^{\text{out}}\langle r_1 p_1 \cdots r_m p_m ; s_1 \overline{p}_1 \cdots s_n \overline{p}_n | \phi'(x) | 0 \rangle = \\
&\int d^{4m}y \; d^{4n}z \; \Big\{ e^{i(\sum p_i \cdot y_i + \sum \overline{p}_j \cdot z_j)} \\
&\prod_{i=1}^{m} \left( \bar{u}_{r_i}(p_i)(\not{p}_i - m) \right)_{\alpha_i} \\
&\langle 0 | T \left( \psi'^{\alpha_1}(y_1) \cdots \psi'^{\alpha_m}(y_m) \overline{\psi}'_{\beta_1}(z_1) \cdots \overline{\psi}'_{\beta_n}(z_n) \phi'(x) \right) | 0 \rangle \\
&\prod_{j=1}^{n} \left( (\not{\overline{p}}_j + m) v_{s_j}(\overline{p}_j) \right)^{\beta_j} \Big\} \; ,
\end{aligned}
$$

following the lines of our derivation of the LSZ reduction formula. You can easily modify this to add scalars to $|\Psi_p\rangle$.

Now, we need to compute $\sigma(k^2)$ to order $\mathcal{O}(g^2)$. Thus we need our Green's functions only to order $\mathcal{O}(g)$, since they are squared to produce $\sigma$. In our theory, only one Green's function is nonzero at this order – the interaction vertex, which at this order is simply given by the tree diagram. Thus, at this order,

$$
\begin{aligned}
&\langle 0 | T \left( \psi'(y) \overline{\psi}'(z) \phi'(x) \right) | 0 \rangle = \\
&\quad = \int \frac{d^4p}{(2\pi)^4} \frac{d^4\overline{p}}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} e^{i(py + \overline{p}z + kx)} \frac{i}{k^2 - \mu^2 + i\epsilon} \\
&\quad \frac{i}{\not{p} - m + i\epsilon} (-ig\Gamma) \frac{i}{\not{\overline{p}} + m + i\epsilon} (2\pi)^4 \delta^{(4)}(k + p + \overline{p}) \; .
\end{aligned}
$$

Plugging this in, we find

$$
\begin{aligned}
&{}^{\text{out}}\langle rp \, ; s\overline{p} | \phi'(0) | 0 \rangle = \frac{i}{(p + \overline{p})^2 - \mu^2} \\
&\bar{u}_r(p)(\not{p} - m) \frac{i}{\not{p} - m + i\epsilon} (-ig\Gamma) \frac{i}{\not{\overline{p}} + m + i\epsilon} (\not{\overline{p}} + m) v_s(\overline{p}) \\
&= -\frac{g}{(p + \overline{p})^2 - \mu^2} \, \bar{u}_r(p) \Gamma v_s(\overline{p}) \; .
\end{aligned}
$$

Returning to our expression for $\sigma(k^2)$, we see that the sum reduces at this order to an integral over on-shell momenta and spin states of the square of this expression. The spin sum here yields

$$
\begin{aligned}
\sum_{rs} |\bar{u}_r(p) \Gamma v_s(\overline{p})|^2 &= \mathrm{Tr} \left( \bar{u}_r(p) \Gamma v_s(\overline{p}) \bar{v}_s(\overline{p}) \Gamma u_r(p) \right) \\
&= \mathrm{Tr} \left( (\not{p} + m) \Gamma (\not{\overline{p}} - m) \Gamma \right) = 4(p \cdot \overline{p} \mp m^2) \; ,
\end{aligned}
$$

where the two signs correspond to $\Gamma = 1$, $i\gamma_5$ respectively. Further, we have here

$$p \cdot \overline{p} = 1/2\left((p+\overline{p})^2 - p^2 - \overline{p}^2\right) = 1/2(k^2 - 2m^2) \ ,$$

so that

$$\sum_{rs} |\bar{u}_r(p)\Gamma v_s(\overline{p})|^2 = 4(p \cdot \overline{p} \mp m^2) = 2k^2 - 4(\alpha - 1)m^2 \ .$$

We perform the integral in the center of mass of this state, in which the total energy is $E_T^2 = (p+\overline{p})^2 = k^2$, where it reduces to the kind described in great detail in the solutions to Problem Set 1.

$$\begin{aligned}
\sigma(k^2)\theta(k^0) = (2\pi)^3 \int & \frac{d^3\mathbf{p}}{(2\pi)^3(2E_{\mathbf{p}})} \frac{d^3\overline{\mathbf{p}}}{(2\pi)^3(2E_{\overline{\mathbf{p}}})} \frac{g^2}{(k^2-\mu^2+i\epsilon)^2} \\
& 4(p\cdot\overline{p} \mp m^2)\delta^{(4)}(p+\overline{p}+k) \\
& = \frac{g^2}{(k^2-\mu^2)^2}\left(2k^2 - 4(\alpha-1)m^2\right)\frac{1}{8\pi^2}\frac{|\mathbf{p}|}{E_T} \\
& = -\frac{g^2}{4\pi^2}\left(k^2-\mu^2\right)^{-2}\sqrt{1/4 - m^2/k^2}\left((2(\alpha-1)m^2 - k^2\right) \ .
\end{aligned}$$

In the last step I used energy conservation to find that in the center of mass frame, $|\mathbf{p}|^2 = k^2/4 - m^2$. Since this is already quadratic in $g$, I can to this order replace $(k^2-\mu^2)^{-2}$ by $|D'(k^2)|^2$, from which it differs by higher-order terms. Thus we have

$$\sigma(k^2) = -|D'(k^2)|^2 \frac{g^2\sqrt{1/4 - m^2/k^2}}{4\pi^2}\left((2(\alpha-1)m^2 - k^2\right) \ ,$$

in agreement with our previous results.