Physics 342

Problem set 3

Due 11/14/04

1. As discussed in class, Coulomb gauge is defined by the gauge-fixing condition

$$\nabla \cdot \mathbf{A} = 0 \ .$$

Show that, in this gauge, the ij part of the photon propagator is

$$-i \frac{g_{ij} + k_i k_j / |\mathbf{k}|^2}{k^2 + i\epsilon} \ ,$$

where $\mathbf{k}$ is the three-momentum. Compute the $i0$ and 00 propagators.

2. A Dirac field, the electron, is minimally coupled to the electrodynamic field with coupling constant e. Compute to order $\mathcal{O}(e^2)$ the Feynman amplitude for electron-electron scattering in both Coulomb and Feynman gauge, and show that the final answers agree when the electron wavefunctions satisfy the Dirac equation.

3. Even in QED, it is possible to work in a gauge in which ghost fields are needed. For example, this is a valid form of the QED Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\text{em}} - \frac{\lambda}{2} \left(\partial_\mu A^\mu + \sigma A_\mu A^\mu \right)^2 + \mathcal{L}_{\text{ghost}} \ .$$

Here $\mathcal{L}_{\text{em}}$ is the standard Lagrangian (without gauge-fixing or ghost terms), and λ, σ are arbitrary real numbers. What is $\mathcal{L}_{\text{ghost}}$? What is the ghost propagator? What are the vertices involving the ghost fields?

4. In class, we showed that as a consequence of the gauge-invariance of QED, renormalization of the photon propagator requires only a wavefunction renormalization couterterm (proportional to $F_{\mu\nu}F^{\mu\nu}$). There is no need for a mass renormalization counterterm proportional to $A_\mu A^\mu$. Verify this by computing to $\mathcal{O}(e^2)$ the renormalized 1PI two-photon Green's function (the inverse propagator) in the theory of a Dirac particle minimally coupled to electromagnetism. Fix the counterterm by using the BPH procedure, ie choose it to cancel the second-order terms in the expansion of the relevant graph about $k = 0$, where k is the photon momentum. Express the

answer as an integral over a Feynman parameter. Use the Pauli-Villars cutofff procedure: subtract from the Fermi loop integral identical loop integrals with heavy masses, with coefficients chosen to cancel both the quadratic and the logarithmic divergence of the integral. Of course, the final result is independent of the heavy masses as they go to infinity. However, show that even at finite heavy masses, the Green's function is proportional to

$$g_{\mu\nu}k^2 - k_\mu k_\nu \ .$$

We will soon see that this is in fact true to all orders. *Historical Note:* This problem was a famous technical headache in the late 40's. If you just blithely manipulate divergent integrals, it looks like a photon mass couterterm *is* needed. Pauli and Villars invented their gauge-invariant cutoff to show that this apparent contradiction to gauge invariance is just an error, not a sign of some deep sickness in the theory.

5. Do the same calculation for a charged scalar replacing the charged Dirac particle, but this time using dimensional regularization. Note that in d dimensions, $g^\mu_\mu = d$.