1. We begin with the potential found using the method of images. 

\[ \Phi(x) = kq \left( \frac{1}{|x-x'|} - \frac{1}{|x-x''|} \right), \]

where if we set the real charge at \( x' = (0,0,z) \), the image charge lies at \( \tilde{x}' = (0,0,-z) \).

(a) To find the surface charge density induced on the conducting plane, we use the fact that at the surface of a conductor

\[ 4\pi k \sigma = E = -\frac{\partial \Phi}{\partial n} \]

and working in cylindrical coordinates we have

\[ 4\pi k \sigma = -\frac{2kqz'}{(r^2 + z'^2)^{3/2}}. \]

(b) Since the electric field created by the induced charge on the plane is the same as the field of the image charge, the force applied by the plane to the charge is simply

\[ F = -\frac{kq^2}{(2z')^2} \hat{z}. \]

The force is attractive.

(c) The force applied by the charge to the plane is

\[ F = \int d^2x \sigma(x) E(x), \]

where of course \( E \) should be the field created by the charge itself. By symmetry, the total force acts normal to the plane and we can restrict attention to this component (a pressure). Furthermore, this normal component is equal to exactly one-half the magnitude of the full field computed above in (a). Thus

\[ F = \int d^2x 2\pi k \sigma^2 \hat{z}, \]
and plugging in our result from (a) we have for the magnitude of this

\[ F = k \int_0^\infty r dr \left( 2\pi \sigma(r) \right)^2 = kq^2 z'^2 \int_0^\infty \frac{r dr}{(r^2 + z'^2)^{3/2}} = \frac{kq^2}{(2z')^2}. \]

(d) To move the charge out to infinity against the attractive force computed above requires work given by

\[ W = -\int_{z'}^\infty F(z) \cdot dz = kq^2 \int_{z'}^\infty \frac{dz}{4z^2} = \frac{kq^2}{4z'}. \]

(e) The potential energy of two charges of equal and opposite magnitude at distance 2z' is

\[ U = -\frac{kq^2}{2z'}, \]

which is precisely twice our result above. The factor of two may be interpreted by noting that if we integrate by parts to express the potential energy as residing in the electric field, we would find that in the image problem precisely one-half of this resides in the half-space discussed in the problem (the field in the other half-space vanishes in the problem at hand).

2. The Dirichlet Green’s function \( G_D(x, x') \), as we have said, is precisely the potential generated at \( x' \) by a point charge of magnitude \( 1/k \) at \( x \) with boundary conditions requiring that the potential vanish along the boundary (as a function of \( x' \)).

(a) Specifically, in our case,

\[ G_D(x, x') = \frac{1}{|x' - x|} - \frac{1}{|x' - \tilde{x}|}, \]

where \( \tilde{x} \) is the reflection of \( x \) in the \( xy \) plane. We can write this explicitly as

\[ G_D(x, x') = \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} - \frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z + z')^2}}. \]

I may have caused some confusion when, in class, I said that \( G_D(x, x') \) is the potential at \( x \) given a charge at \( x' \) and vanishes for \( x \) on the boundary. This is not too bad, however, because in fact

\[ G_D(x, x') = G_D(x', x). \]
The explicit form above clearly demonstrates this. To see that it holds in general, use Green’s theorem with \( \phi(y) = G_D(x, y) \) and \( \psi(y) = G_D(x', y) \) to get
\[
\int_V d^3y \left[ G_D(x, y) \delta^{(3)}(x' - y) - G_D(x', y) \delta^{(3)}(x - y) \right] = G_D(x, x') - G_D(x', x) = 0.
\]

(b) Our expression for the solution of the Dirichlet problem is
\[
\Phi(x) = k \int_V d^3x' \rho(x') G_D(x, x') - \frac{1}{4\pi} \int_{\partial V} d^2x' \Phi(x') \frac{\partial G_D(x, x')}{\partial n'}.
\]
In the case at hand, \( \rho = 0 \) and the integral we need to perform is the surface integral only. The normal derivative was computed in Problem 1 above, so we have explicitly (recalling the appropriate sign for the outward normal)
\[
\Phi(x) = \frac{1}{4\pi} \int_{z'=0} d^2x' \Phi(x') \frac{2z}{((x - x')^2 + (y - y')^2 + z^2)^{3/2}}.
\]
Inserting the boundary potential given, we work in cylindrical coordinates and find (Because of the way part (d) of this problem works, I am going to change my notation and - just this once! - use \( \rho \) for the distance from the \( z \) axis. Since we have already stated that this problem includes no free charges, no confusion as to the meaning of \( \rho \) should arise.)
\[
\Phi(\rho, \theta, z) = \frac{V}{2\pi} \int_0^{2\pi} d\theta' \int_0^a \rho' d\rho' \frac{2z}{(\rho^2 - 2\rho \rho' \cos(\theta - \theta') + \rho^2 + z^2)^{3/2}}.
\]
A shift in \( \theta' \) shows that the result is independent of \( \theta \) as is expected by symmetry
\[
\Phi(\rho, \theta, z) = \frac{V}{2\pi} \int_0^{2\pi} d\theta' \int_0^a \rho' d\rho' \frac{z}{(\rho^2 - 2\rho \rho' \cos \theta' + \rho^2 + z^2)^{3/2}}.
\]
(c) Setting \( \rho = 0 \) we have
\[
\Phi(0, 0, z) = Vz \int_0^a \frac{\rho' d\rho'}{\rho^2 + z^2} = V \left( 1 - \frac{1}{\sqrt{a^2 + z^2}} \right).
\]
(d) In this problem we will mix our notation slightly, using both \( r = |x| \) and \( z \), so that \( r^2 = \rho^2 + z^2 \), and write the potential in a mixed notation convenient for this problem as a function of \( r, \theta, z \)
\[
\Phi(r, \theta, z) = \frac{V}{2\pi} \int_0^{2\pi} d\theta' \int_0^a \rho' d\rho' \frac{z}{(\rho^2 - 2\rho \rho' \cos \theta' \sqrt{r^2 - z^2} + r^2)^{3/2}}.
\]
We are asked to perform an expansion for \( r \gg a \). Inspecting the integrand, we see that this expansion is a bit ambiguous. We can write

\[
\frac{z}{(\rho^2 - 2\rho \cos \theta' \sqrt{r^2 - z^2} + r^2)^{3/2}} = \frac{z}{r^3} \left(1 - \frac{2\rho' \cos \theta' \sqrt{r^2 - z^2}}{r^2} + \left(\frac{\rho'}{r}\right)^2\right)^{-3/2}.
\]

How the second term scales with \( r \) depends on the direction in which we go to large-\( r \). Maintaining a finite angle from the \( z \) axis, we have \( \rho = \sqrt{r^2 - z^2} \sim r \) and this term is of order \( 1/r \). On the other hand, maintaining fixed \( \rho \) as \( r \to \infty \), this term is of order \( 1/r^2 \). I will not assume this, so will take this term to be of order \( 1/r \). Before expanding, note that when we perform the \( \theta' \) integration, odd powers of \( \cos \theta' \) integrate to zero. So I will drop these from my expansion and write the even terms only

\[
\frac{z}{(r^2 - 2r \cos \theta' \sqrt{r^2 - z^2} + r^2)^{3/2}} \sim \frac{z}{r^3} \left[1 - \frac{3}{2} \left(\frac{\rho'}{r}\right)^2 + \frac{15\rho'^2 \cos^2 \theta' (r^2 - z^2)}{2r^4} + \frac{15}{8} \left(\frac{\rho'}{r}\right)^4 - \frac{105r'^4 \cos^2 \theta (r^2 - z^2)}{4r^6} + \cdots\right],
\]

where terms of order \((\rho'/r)^{-6}\) have been dropped, and \( \rho \) is considered of the same magnitude as \( r \). Integrating, we set \( \cos^2 \theta' \to 1/2 \) and perform the \( \rho' \) integration to find

\[
\Phi(r, \theta, z) \sim \frac{Vza^2}{2r^3} \left[1 - \frac{3}{4} \left(\frac{a}{r}\right)^2 + \frac{15a^2(r^2 - z^2)}{8r^4} + \frac{5}{8} \left(\frac{a}{r}\right)^4 - \frac{35a^4(r^2 - z^2)}{8r^6} + \cdots\right].
\]

Setting \( r = z \) here (clearly here \( \rho = 0 \) and our concerns above are not an issue) we find

\[
\Phi(z = r) \sim \frac{Va^2}{2z^2} \left[1 - \frac{3}{4} \left(\frac{a}{z}\right)^2 + \frac{5}{8} \left(\frac{a}{z}\right)^4\right],
\]

which is precisely the expansion of (c) in powers of \( a/z \) to this order.

3. The potential due to a line charge \( \mu \) at position \( x' \) in the \( xy \) plane and parallel to the \( z \) axis is given by

\[
\Phi(x) = 2k\mu \ln\left(||x - x'||/r_0\right),
\]

where all vectors in this problem are two-dimensional, and \( r_0 \) is arbitrary, serving to set the zero of the potential.
(a) Adding an image line charge of density \( \tilde{\mu} \) at position (in the plane) \( \tilde{x}' \) we have

\[
\Phi(x) = 2k (\mu \ln(|x - x'|) + \tilde{\mu} \ln(|x - \tilde{x}'|)) - C ,
\]

where I have chosen a more direct parameterization of the additive constant. The equipotential surfaces satisfy

\[
|x - x'|^\mu = N|x - \tilde{x}'|^{-\tilde{\mu}} ,
\]

where the constant \( N = e^{(\Phi + C)/2k} \). With \( \tilde{\mu} = -\mu \) we have equipotential surfaces given by

\[
|x - x'| = N^{1/\mu}|x - \tilde{x}'| .
\]

The set of points whose distances from two distinct points are in a fixed ratio forms a circle. We’ll need this in the next problem so let’s show it now. Set

\[
(x - x')^2 + (y - y')^2 = K ((x - \tilde{x}')^2 + (y - \tilde{y}')^2) ,
\]

and move things around a bit to find that this describes a circle centered at

\[
\left( \frac{K\tilde{x}' - x'}{K-1}, \frac{K\tilde{y}' - y'}{K-1} \right)
\]

with radius given by

\[
r^2 = \frac{K}{(K-1)^2} \left( (x' - \tilde{x}')^2 + (y' - \tilde{y}')^2 \right) .
\]

In our case, the constant \( K = N^{2/\mu} = e^{(\Phi + C)/k\mu} \). To meet our boundary conditions, we need \( \Phi = 0 \) along a circle of radius \( a \), which we will center around the origin. We place the real charge \( \mu \) at \( x' = (d, 0) \) and find

\[
K = e^{C/k\mu} \\
\tilde{x}' = (d/K, 0) \\
a^2 = \frac{K}{(K-1)^2} (d - d/K)^2 .
\]
Solving these, we find

\[ K = \frac{d^2}{a^2} \]

\[ \tilde{x}' = \left( \frac{a^2}{d}, 0 \right) \]

\[ C = k\mu \ln \left( \frac{d^2}{a^2} \right) . \]

(b) Inserting this into our expression for the potential we have finally

\[ \Phi(x) = 2k\mu \ln \left( \frac{a|x - x'|}{d|x - \tilde{x}'|} \right) = k\mu \ln \left( \frac{a^2(r^2 + d^2 - 2rd\cos\theta)}{d^2r^2 + a^4 - 2a^2rd\cos\theta} \right) . \]

For \( r \gg d > a \), we have

\[ \Phi \sim 2k\mu \left( \ln(a/d) - \frac{d}{r} \frac{1}{1 - (a/d)^2} \cos\theta \right) + O(r^{-2}) . \]

(c) The surface charge density at \((a, \theta)\) is given as usual by

\[ 4\pi k\sigma(\theta) = -\frac{\partial\Phi(r, \theta)}{\partial r} \bigg|_{r=a} \]

and some calculation leads to

\[ \sigma(\theta) = -\frac{\mu}{2\pi a} \frac{1}{1 + (a/d)^2 - 2(a/d)\cos\theta} . \]

(d) The force per unit length on the line charge is directed toward the center of the cylinder and has magnitude (given by computing the force applied by the image charge)

\[ |\mathbf{F}| = \frac{2k\mu^2}{d(1 - (a/d)^2)} . \]

I noted that this problem may be simplified using conformal mapping techniques. Let’s see how that would go. An important question in this regard is how to deal with charge densities. The Laplacian is not preserved under conformal maps, but it transforms like a density. If \( \Phi = \Re(F(z)) \), \( z = x + iy \), and \( z(w) \) is an analytic function with \( w = u + iv \), we have

\[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = |z'(w)|^{-2} \left( \frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} \right) , \]
where we use the conformal map to consider \( \Phi \) a function of \((u, v)\). This means that we should require charges to transform nontrivially. Fortunately, charge densities naturally transform in this way. In particular,

\[
\delta^{(2)}(z(w) - z(w_0)) = |z'(w)|^{-2}\delta^{(2)}(w - w_0) .
\]

This allows us to map our problem to a simpler one. In the case at hand, the boundary problem \( \Phi(z) = 0 \) for \(|z| = a\) with a line charge \( \mu \) at \( z = d \) is mapped by

\[
w = i\frac{z - ia}{z + ia}
\]

with inverse

\[
z = -ia\frac{w + i}{w - i}
\]
to the problem of a line charge \( \mu \) located at \( w' = i(d - ia)/(d + ia) \) with potential vanishing along the image of the circle, which is the real line \( w = \bar{w} \). The solution in this case is simply obtained by placing an image charge at \( \tilde{w}' = \bar{w}' \). Check that this corresponds to \( z = a^2/d \). The potential is then the real part of

\[
F(w(z)) = 2k\mu \ln \left( \frac{w - w'}{w - \bar{w}'} \right)
\]

\[
= 2k\mu \ln \left[ \frac{(i\frac{z - ia}{z + ia} - i\frac{d - ia}{d + ia})}{i\left(\frac{z - ia}{z + ia} + i\frac{d + ia}{d - ia}\right)} \right]
\]

\[
= 2k\mu \ln \left( \frac{ia(z - d)(d - ia)}{(zd - a^2)(d + ia)} \right) .
\]

Taking the real part reproduces the result above.

4.

(a) We essentially solved this above. Setting the potential at infinity to zero and (for convenience) locating our charges along the \( x \)-axis at points \( x' \) and \( \tilde{x}' = x' - R \), we have as the equipotential surface with potential \( V \) a circle centered at

\[
\left( \frac{K\tilde{x}' - x}{K - 1}, 0 \right)
\]

with

\[
K = e^{V/k\mu} .
\]
the radius is given by 
\[ r^2 = \frac{K}{(K-1)^2 R^2}. \]

(b) To find the capacitance we place a charge per unit length \( \mu \) on one conductor and \( -\mu \) on the other, and compute the potential difference between the two. Using the results of (a) we can reproduce the potential everywhere outside the two conducting surfaces by replacing each conductor with a line charge density, positioned so that the two cylinders are each an equipotential. If the cylinder of radius \( a \) is at potential \( V_a \) and the other at potential \( V_b \), and if \( K(V) \) is as above, the following must hold

\[
\frac{K_a x' - x}{K_a - 1} - \frac{K_b x' - x}{K_b - 1} = d
\]
\[
\frac{K_a}{(K_a - 1)^2 R^2} = a^2
\]
\[
\frac{K_b}{(K_b - 1)^2 R^2} = b^2
\]

Computing, we find that this implies (recalling here that with \( \Phi = 0 \) at infinity we have \( (K_a - 1)(K_b - 1) < 0 \))

\[
d^2 - a^2 - b^2 = \frac{K_a + K_b}{(K_a - 1)(K_b - 1)}
\]
\[
2ab = \frac{2\sqrt{K_a K_b}}{(K_a - 1)(K_b - 1)},
\]

whence

\[
\cosh^{-1}\left( \frac{d^2 - a^2 - b^2}{2ab} \right) = \frac{1}{2} \ln\left( \frac{K_a}{K_b} \right) = \frac{V_a - V_b}{2k\mu}.
\]

Setting \( C^{-1} = (V_a - V_b)/\mu \) we have

\[
C^{-1} = 2k \cosh^{-1}\left( \frac{d^2 - a^2 - b^2}{2ab} \right).
\]

(c) Agreement with our previous effort at this problem, in which we neglected the variation of the potential over the conductors, is expected in the limit \( d >> a, b \). Indeed, in this limit we have

\[
C^{-1} \sim 2k \left[ \ln\left( \frac{d^2}{ab} \right) - \frac{a^2 + b^2}{d^2} + \cdots \right],
\]
where the expansion is in powers of $a/d$ and $b/d$.

5. The Dirichlet Green’s function $G_D(x, x')$ can be found, as we have seen in the previous Problem Set, as the potential at the point $x$ in the presence of a point (line, in our case) charge of magnitude (line density) $1/k$ located at $x'$ and subject to vanishing Dirichlet boundary conditions, i.e. vanishing for $x \in \partial V$. In class, we saw how the conformal map

\[ z = w^{\pi/\beta} \]

inverted by

\[ w = z^{\beta/\pi} \]

maps this problem to the upper half-plane with vanishing Dirichlet boundary conditions. As discussed above, the charge maps simply, so to find the Green’s function for the corner we need only find the potential for a point (line) charge in the half plane (space); the position $w'$ of this is found from the position $z'$ of our line charge using the inverse map. This, in turn, is solved by an image charge of equal and opposite magnitude at $\bar{w}'$, which leads to

\[ F(w) = 2 \ln \left( \frac{w - w'}{w - \bar{w}'} \right) . \]

The conformal map now yields

\[ F(z) = F(w(z)) = 2 \ln \left( \frac{z^{\pi/\beta} - z'^{\pi/\beta}}{z^{\pi/\beta} - (\bar{z}')^{\pi/\beta}} \right) . \]

The Green’s function is now found as the real part of this

\[ G_D(x, x') = \ln \left( \frac{r^{2\pi/\beta} + r'^{2\pi/\beta} - (rr')^{\pi/\beta} \cos \left( \frac{\pi(\theta - \theta')}{\beta} \right)}{r^{2\pi/\beta} + r'^{2\pi/\beta} - (rr')^{\pi/\beta} \cos \left( \frac{\pi(\theta + \theta')}{\beta} \right)} \right) . \]