1. We are to compute the Dirichlet Green’s function for a volume $V$ consisting (in cylindrical coordinates) of the points $(r, \phi, z)$ with $r < a$ and $0 < z < L$. The Green’s function will be a function of $x$ satisfying

$$\nabla^2 G_D(x, x') = -4\pi \delta^{(3)}(x - x')$$

$$= -\frac{4\pi}{r} \delta(r - r') \delta(z - z') \delta(\phi - \phi')$$

$$G_D(x, x') = 0 \quad r = a$$

$$G_D(x, x') = 0 \quad z = 0, L.$$ 

The solution to the problem will be given by $kqG_N$. Note that to respect the problem’s notation I have slightly modified the way boundary conditions are imposed on $G$ relative to our class conventions. Since $G_D$ is symmetric in its arguments, this is unimportant anyway.

To get a first form for $G_D$ write the $\delta$ functions for $\phi$ and $r$ as

$$\delta(\phi - \phi') = \frac{1}{2\pi} \sum_{m=\pm\infty} e^{im(\phi - \phi')}$$

$$\frac{1}{r} \delta(r - r') = \sum_{n=1}^{\infty} \frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})} J_\nu(x_{\nu n} r/a) J_\nu(x_{\nu n} r'/a).$$

The latter equation holds for any $\nu \geq -1$. It is obtained from eqns. (4.13) and (4.14) in the notes by setting $f(r) = \delta(r - r')$.

Similarly expanding $G_D$ (recall this is a complete set of functions satisfying the boundary conditions in $r$)

$$G_D(x, x') = \sum_{m=\pm\infty} e^{im(\phi - \phi')} \sum_{n=1}^{\infty} J_m(x_{mn} r/a) f_{mn}(r'; z, z'),$$

and inserting into the Poisson equation, we find that

$$f_{mn}(r'; z, z') = \frac{2}{a^2 J_{\nu+1}^2(x_{\nu n})} J_m(x_{mn} r'/a) g_{mn}(z, z').$$
where the vertical function $g_{mn}(z, z')$ satisfies
\[
\left[ \frac{\partial^2}{\partial z^2} - \left(\frac{x_{mn}}{a}\right)^2 \right] g_{mn}(z, z') = -2\delta(z - z') .
\]

For $z \neq z'$ this is easily solved by
\[
g_{mn}(z, z') = A_{mn}(z') \sinh \left( \frac{x_{mn}}{a} z \right) + B_{mn}(z') \cosh \left( \frac{x_{mn}}{a} z \right) .
\]

Imposing the boundary conditions at $z = 0, L$ we have
\[
g_{mn}(z, z') = \begin{cases} 
A_{mn}(z') \sinh \left( \frac{x_{mn}}{a} z \right) & z < z' \\
A_{mn}'(z') \sinh \left( \frac{x_{mn}}{a} (L - z) \right) & z > z' 
\end{cases} .
\]

The vertical function must be continuous at $z = z'$. Imposing this we see that
\[
g_{mn}(z, z') = A_{mn} \sinh \left( \frac{x_{mn}}{a} z < \right) \sinh \left( \frac{x_{mn}}{a} (L - z > \right) ,
\]

where, as usual,
\[
z_> = \left\{ \min_{\max} (z, z') \right\} .
\]

Inserting this back into the equation and integrating through $z = z'$ we find that we must have
\[
-A_{mn} \frac{x_{mn}}{a} \sinh \left( \frac{x_{mn}}{a} L \right) = -2 .
\]

Putting all of this together we have our first form
\[
G_D(x, x') = \frac{4}{a} \sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} \sum_{n=1}^{\infty} J_m(x_{mn} r'/a) J_m(x_{mn} r/a) \frac{1}{\nu+1} (x_{\nu n} \sinh (\frac{x_{mn} L}{a})) \times \sinh \left( \frac{x_{mn}}{a} z < \right) \sinh \left( \frac{x_{mn}}{a} (L - z > \right) .
\]

To get a second form, we exchange the roles of $z$ and $r$ in the above. Thus we expand the $\delta$ function in $\phi$ but also in $z$ using
\[
\delta(z - z') = \frac{2}{L} \sum_{n=1}^{\infty} \sin(n \pi z/L) \sin(n \pi z'/L) .
\]
We write $G_D$ in a complete set of functions satisfying the boundary conditions in $z$ as

$$G_D(x, x') = \sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} \sum_{n=1}^{\infty} \sin(n\pi z/L) f_{mn}(z'; r, r') .$$

Inserting into the Poisson equation we find

$$f_{mn}(z'; r, r') = \frac{2}{L} \sin(n\pi z'/L) g_{mn}(r, r') ,$$

where the radial function $g_{mn}(z, z')$ satisfies

$$\left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \left( \frac{n\pi}{L} \right)^2 + \frac{m^2}{r^2} \right] g_{mn}(r, r') = -\frac{2}{r} \delta(r - r') .$$

For $r \neq r'$ this is solved by

$$g_{mn}(r, r') = A_{mn}(r') I_m\left(n\pi r/L\right) + B_{mn}(r') K_m\left(n\pi r/L\right) .$$

In order for the solution to be regular at the origin, and to vanish at $r = a$, we have

$$g_{mn}(r, r') = \begin{cases} A_{mn}(r') I_m\left(n\pi r/L\right) & r < r' \\ A_{mn}'(r') \left(I_m\left(n\pi a/L\right) K_m\left(n\pi r'/L\right) - K_m\left(n\pi a/L\right) I_m\left(n\pi r'/L\right)\right) & r > r' \end{cases} .$$

We want the radial function to be continuous at $r = r'$, hence

$$g_{mn}(r, r') = A_{mn} I_m\left(n\pi r/L\right) \left[I_m\left(n\pi a/L\right) K_m\left(n\pi r'/L\right) - K_m\left(n\pi a/L\right) I_m\left(n\pi r'/L\right)\right] ,$$

where, as always,

$$r_\geq = \left\{ \min \left( r, r' \right) \right\} ,$$

$$r_\leq = \left\{ \max \left( r, r' \right) \right\} .$$

Finally, integrating the radial equation through the singularity at $r = r$ we find

$$A_{mn} \frac{n\pi}{L} I(n\pi a/L) \left[I_m\left(n\pi r'/L\right) K'_m\left(n\pi r'/L\right) - I'_m\left(n\pi r'/L\right) K_m\left(n\pi r'/L\right)\right]$$

$$= -A_{mn} \frac{1}{r'} I(n\pi a/L) = -\frac{2}{r'} ,$$

where I have used Jackson’s equation (3.147) computing the Wronskian. Putting it all together, we have a second form

$$G_D(x, x') = \frac{4}{L} \sum_{m=-\infty}^{\infty} e^{im(\phi - \phi')} \sum_{n=1}^{\infty} \sin(n\pi z/L) \sin(n\pi z'/L) \frac{I_m\left(n\pi r_\leq/L\right)}{I_m\left(n\pi a/L\right)}$$

$$\times \left[I_m\left(n\pi a/L\right) K_m\left(n\pi r_\geq/L\right) - K_m\left(n\pi a/L\right) I_m\left(n\pi r_\geq/L\right)\right] .$$
The last form follows by expanding $G_D$ in eigenfunctions of the Laplacian with the Dirichlet boundary conditions. The uniqueness theorem guarantees that there is no zero eigenvalue, and our basis of eigenfunctions may be taken to be

$$\psi_{mkn}(r, \phi, z) = e^{im\phi} \sin(k\pi z/L)J_m(x_{mn}r/a),$$

with eigenvalue

$$\nabla^2 \psi_{mkn} = \left[ \left( \frac{x_{mn}}{a} \right)^2 + \left( \frac{n\pi}{L} \right)^2 \right] \psi_{mkn},$$

and normalized to

$$\int_0^{2\pi} d\phi \int_0^L dz \int_0^a dr \psi^*_{m'k'n'}(r, \phi, z) \psi_{mkn}(r, \phi, z) = \frac{a^2 L \pi J_{m+1}^2(x_{mn})}{2} \delta_{mm'} \delta_{kk'} \delta_{nn'}.$$

Now we simply plug these into eqn. (2.36) from the second week’s notes to find our third form

$$G_D(x, x') = \frac{8}{La^2} \sum_{m=-\infty}^{\infty} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} e^{im(\phi-\phi')} \frac{\sin(k\pi z/L) \sin(k\pi z'/L) J_m(x_{mn}r/a) J_m(x_{mn}r'/a)}{\left( \frac{x_{mn}}{a} \right)^2 + \left( \frac{k\pi}{L} \right)^2} J_{m+1}^2(x_{mn}).$$

The relation between the different approaches was discussed in class. In the third form, the inhomogeneous equation is solved using the eigenfunction expansion. This means essentially that we expand the $\delta$ functions in all three variables. The cost is an extra sum. The advantage is that there is no need to treat one variable as an exception and deal with piecewise-defined functions.

2. Now, to use this for the problem at hand we need to plug the various forms computed above into our equation (1.58). Since $\rho = 0$ we have no volume integral. In the surface integral, the only contribution is from the disc $r \leq b < a$ at $z = L$. On this boundary, the normal to the surface is $\hat{z}$ so (1.58) takes the form

$$\Phi(x) = -\frac{V}{4\pi} \int_0^{2\pi} d\phi' \int_0^b r'dr' \frac{\partial G_D(x; r', \phi', L)}{\partial z'}.$$

The angular integration will yield the same result in all cases, restricting the sum over $m$ to $m = 0$. The azimuthal symmetry simplifies things as usual.
(a) Getting the expansions is now a matter of plugging and performing the radial integral. It will be useful to note that

\[ A_n = \int_0^b r' dr' J_0(x_0 r'/a) = \frac{ab}{x_0} J_1(x_0 b/a) . \]

We find from the first form

\[ \Phi(r, \phi, z) = \frac{2V}{a^2} \sum_{n=1}^{\infty} A_n J_0(x_0 r/a) \sinh(x_0 z/a) J_2^2(x_0) \sinh(x_0 L/a) . \]

For the second form we will need to work a little bit harder. We will need

\[ B_n(r) = \int_0^r r' dr' I_0(n \pi r'/L) = \frac{L r}{n \pi} I_1(n \pi r/L) \]

\[ C_n(r) = \int_r^b r' dr' K_0(n \pi r'/L) = \frac{L}{n \pi} [r K_1(n \pi r/L) - b K_1(n \pi b/L)] , \]

for \(0 < r \leq b\), and distinguish the case \(r > b\) where we have

\[ \Phi(r, \phi, z) = \frac{2 \pi V}{L^2} \sum_{n=1}^{\infty} \left( -1 \right)^n n \sin(n \pi z/L) B_n(b) \]

\[ \times \left[ I_m(n \pi a/L) K_m(n \pi r/L) - K_m(n \pi a/L) I_m(n \pi r/L) \right] . \]

For \(r < b\) we have instead

\[ \Phi(r, \phi, z) = -\frac{2 \pi V}{L^2} \sum_{n=1}^{\infty} \left( -1 \right)^n n \sin(n \pi z/L) \]

\[ \times \left[ B_n(r) [I_0(n \pi a/L) K_0(n \pi r/L) - K_0(n \pi a/L) I_0(n \pi r/L)] \right. \]

\[ + \left. I_0(n \pi r/L) [I_0(n \pi a/L) C_n(r) - K_0(n \pi a/L) (B_n(b) - B_n(r))] \right] . \]

For the third form we find

\[ \Phi(r, \phi, z) = -\frac{2V}{L^2 a^2} \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left[ \frac{(-1)^k \sin(k \pi z/L) J_0(x_0 r/a) A_n}{(x_0 a)^2 + \left( \frac{k \pi \alpha}{L} \right)^2} \right] J_1^2(x_0) . \]

(b) We substitute \(z = a = 2b = L/2\) and \(r = 0\) above, and compute numerically. The first form becomes

\[ \Phi(0, \phi, L/2) = V \sum_{n=1}^{\infty} \frac{J_1(x_0 n/2) \sinh(x_0 n)}{x_0 n J_1^2(x_0 n) \sinh(2x_0 n)} . \]
The second form becomes

\[ \Phi(0, \phi, L/2) = \frac{V}{2} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi/2)}{I_0(n\pi/2)} \times [I_0(n\pi/2)K_1(n\pi/4) + K_0(n\pi/2)I_1(n\pi/4)] \] .

The third form becomes

\[ \Phi(0, \phi, L/2) = 2\pi V \sum_{l=0}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^l(2l+1)J_1(x_0n/2)}{4x_0^2 + (2l+1)^2\pi^2} x_0 J_1^2(x_0n) . \]

3. We have a localized charge density \( \rho(x) \) in an external field described by the potential \( \Phi^{(0)}(x) \), which is slowly varying in the region where \( \rho \) is nonzero.

(a) The total force on the charge distribution is given by

\[ \mathbf{F} = -\int d^3x \rho(x) \nabla \Phi^{(0)}(x) . \]

Using the slow variation of the potential, we expand

\[ \Phi^{(0)}(x) = \Phi^{(0)}(0) + \mathbf{x} \cdot \nabla \Phi^{(0)}(0) + \frac{1}{2} \sum_{ij} x_i x_j \frac{\partial^2 \Phi^{(0)}}{\partial x_i \partial x_j} + \frac{1}{6} \sum_{ijk} x_i x_j x_k \frac{\partial^3 \Phi^{(0)}}{\partial x_i \partial x_j \partial x_k} + \cdots , \]

and then take the derivative to find

\[ -\nabla \Phi^{(0)}(x) = \mathbf{E}^{(0)}(0) + [(\mathbf{x} \cdot \nabla)\mathbf{E}] (0) + \left[ \frac{1}{2} \sum_{jk} x_j x_k \frac{\partial^2 \mathbf{E}^{(0)}}{\partial x_j \partial x_k} \mathbf{E}^{(0)} \right] (0) + \cdots . \]

Inserting this we find

\[ \mathbf{F} = \mathbf{E}^{(0)}(0) \int d^3x \rho(x) + \left( \int d^3x \rho(x) \mathbf{x} \right) \cdot \nabla \mathbf{E}^{(0)}(0) + \frac{1}{2} \sum_{jk} \frac{\partial^2 \mathbf{E}^{(0)}}{\partial x_j \partial x_k} (0) \int d^3x \rho(x) x_j x_k + \cdots . \]

In the second term, we use the fact that \( \nabla \times \mathbf{E} = 0 \) which means that

\[ \frac{\partial E_i}{\partial x_j} = \frac{\partial E_j}{\partial x_i} \]

which lets us write the second term as

\[ (\mathbf{p} \cdot \nabla)\mathbf{E}(0) = \nabla (\mathbf{p} \cdot \mathbf{E})(0) . \]
In the third term, we use the same trick to write
\[
\frac{\partial^2 E_i}{\partial x_j \partial x_k} = \frac{\partial^2 E_j}{\partial x_i \partial x_k},
\]
to find
\[
\frac{1}{6} \sum_{jk} \int d^3x x_j x_k \rho \nabla \frac{\partial E_j^{(0)}}{\partial x_k} = \frac{1}{6} \sum_{jk} \int d^3x (3x_j x_k - r^2 \delta_{jk}) \rho \nabla \frac{\partial E_j^{(0)}}{\partial x_k},
\]
where the last term uses the fact that for the external field, generated by charges outside our volume,
\[
\sum_{jk} \delta_{jk} \frac{\partial E_j^{(0)}}{\partial x_k} = \nabla \cdot \mathbf{E}^{(0)} = 0.
\]

Putting it back together, we have as desired
\[
\mathbf{F} = q\mathbf{E}^{(0)}(0) + \nabla (\mathbf{p} \cdot \mathbf{E})(0) + \frac{1}{6} \sum_{jk} \nabla \left[ Q_{jk} \frac{\partial E_j^{(0)}}{\partial x_k} \right] (0).
\]

Comparing this to the expansion of the energy of a charge distribution in an external field, it does seem as though we have “taken a derivative.” As Jackson points out, the energy expansion eqn. (3.16) in our notes ((4.24) in Jackson) is a functional of \(\rho\) and \(\mathbf{E}^{(0)}\) and not really a function of any space coordinate. To obtain our expression for the force from (3.16) one varies the distribution \(\rho\) by translating it bodily
\[
\rho \to \rho' \quad \rho'(\mathbf{x}) = \rho(\mathbf{x} - \delta \mathbf{x}).
\]

The total force is found as an expansion by varying each term in the expression for the energy, resulting in the expression above.

(b)
\[
N_i = \sum_{jk} \epsilon_{ijk} \int d^3x \rho(\mathbf{x}) x_j E_k(\mathbf{x}).
\]

We insert the expansion of \(\mathbf{E}\) and find
\[
N_i = \sum_{jk} \epsilon_{ijk} E_k(0) \int d^3x x_j \rho(\mathbf{x}) + \sum_{jkl} \epsilon_{ijk} \frac{\partial E_k}{\partial x_l}(0) \int d^3x x_j x_k \rho(\mathbf{x}) + \cdots.
\]
The first term has the desired form already. In the second term we need to use the symmetry of \( \frac{\partial E_k}{\partial x_i} \) and then the vanishing of \( \nabla \cdot E \):

\[
\sum_{jkl} \epsilon_{ijk} \frac{\partial E_k}{\partial x_l}(0) \int d^3 x x_l x_j \rho(x) = \frac{1}{3} \sum_{jkl} \epsilon_{ijk} \frac{\partial E_l}{\partial x_k}(0) \int d^3 x (3x_l x_j - r^2 \delta_{jl}) = \frac{1}{3} \sum_{jkl} \epsilon_{ijk} Q_{lj} \frac{\partial E_l}{\partial x_k}(0) .
\]

Thus

\[
N_i = \sum_{jk} \epsilon_{ijk} p_j E_k(0) + \frac{1}{3} \sum_{jkl} \epsilon_{ijk} Q_{lj} \frac{\partial E_l}{\partial x_k}(0) + \cdots .
\]

4.

(a) We write an expansion of the potential inside as well as outside the sphere, imposing the obvious conditions that the potential is well-behaved at the origin and decays at infinity, and that it has the proper singularity at the position of the charge, which we take to be at \( x = d \hat{z} \) to make the azimuthal symmetry manifest. Then we can expand the potential in Legendre polynomials as

\[
\Phi_{r>a} = \frac{q}{4\pi \epsilon_0} \sum_{l \geq 0} \left[ \frac{r^l}{r^{l+1}} + A_l r^{-l-1} \right] P_l(\cos \theta)
\]

\[
\Phi_{r<a} = \frac{q}{4\pi \epsilon} \sum_{l \geq 0} B_l r^l P_l(\cos \theta) ,
\]

where \( r_{\min} = \{ \min \max (a, d) \} \). The normalization of \( A_l, B_l \) is chosen for convenience. Now we use our matching conditions at \( r = a \)

\[
\frac{\partial \Phi_{r<a}}{\partial \theta} |_{r=a} = \frac{\partial \Phi_{r>a}}{\partial \theta} |_{r=a}
\]

\[
\epsilon \frac{\partial \Phi_{r<a}}{\partial r} |_{r=a} = \epsilon_0 \frac{\partial \Phi_{r>a}}{\partial r} |_{r=a} .
\]

and the orthogonality of \( P_l \) and of their derivatives to get relations on the coefficients

\[
\frac{\epsilon}{\epsilon_0} \left( \frac{a^l}{a^{l+1}} + A_l a^{-l-1} \right) = B_l a^l
\]

\[
\left( l \frac{a^{l-1}}{a^{l+1}} - (l + 1) A_l a^{-l-2} \right) = l B_l a^{l-1}
\]
Solving these, we have in terms of $\kappa = \epsilon / \epsilon_0$

$$A_l = \frac{l(l - 1)}{l + 1 - l\kappa} \frac{a^{2l+1}}{d^{l+1}}$$

$$B_l = \frac{\kappa}{d^{l+1}(l + 1 - l\kappa)} .$$

(b) At small $r$, the leading behavior of the potential is

$$\Phi(r) = \frac{q}{4\pi\epsilon_0} \left( \frac{1}{d} + \frac{r \cos \theta}{d^2(2 - \kappa)} + \frac{r^2(3\cos^2 \theta - 1)}{2d^3(3 - 2\kappa)} + O(r^3) \right) .$$

Thus to leading order

$$\mathbf{E} = -\nabla \Phi = k q \left[-\frac{\hat{z}}{d^2(2 - \kappa)} - \frac{2z\hat{z} - x\hat{x} - y\hat{y}}{d^3(3 - 2\kappa)} + O(r^2) \right] .$$

(c) In the limit $\kappa \to \infty$ we see that $\mathbf{E} \to 0$ as expected.

5.

(a) This problem is simpler than it looks. An Ansatz

$$\mathbf{E} = C \frac{\mathbf{x}}{|\mathbf{x}|^3}$$

for some constant $C$ solves the problem. The continuity of $\mathbf{E}_\parallel$ is clear, while the continuity of $D_\perp$ follows simply from $D_\perp = 0$ on both sides of the interface. The constant $C$ is determined by Gauss’s law

$$\int_S \mathbf{D} \cdot d\mathbf{a} = Q .$$

In our case, $\mathbf{D}$ is everywhere normal to the surface of a sphere at radius $a < r < b$, but its magnitude differs over the two hemispheres. Thus

$$\int_S \mathbf{D} \cdot d\mathbf{a} = 2\pi(\epsilon_0 + \epsilon)C = Q$$

determines $C$ and we have

$$\mathbf{E} = \frac{Q}{2\pi(\epsilon + \epsilon_0)} \frac{\mathbf{x}}{|\mathbf{x}|^3} .$$
(b) Inside the conductor $\mathbf{D} = \mathbf{E} = 0$. Thus the normal component of $\mathbf{D}$ at $r = a$ is

$$\mathbf{D}_\perp = 4\pi \sigma .$$

Once more we compute this separately in each hemisphere and find

$$\sigma = \frac{Q}{2\pi a^2} \left\{ \begin{array}{ll}
\frac{1}{\kappa+1} & \text{empty} \\
\frac{\kappa}{\kappa+1} & \text{dielectric}
\end{array} \right\} ,$$

where again $\kappa = \epsilon/\epsilon_0$.

(c) The polarization in the dielectric is given by

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} = (\epsilon - \epsilon_0) \mathbf{E} .$$

The surface polarization-charge density on the surface of the dielectric at $r = a$ is simply $-P_r$. Thus

$$\sigma_{\text{pol}} = (\epsilon_0 - \epsilon) E_r = -\frac{Q}{2\pi a^2} \frac{\kappa - 1}{\kappa + 1} .$$