6.6. Localized Current Distributions

A full discussion of the multipole expansion for the magnetic field of a current distribution is best conducted in terms of vector spherical harmonics. These are relegated to next semester’s class, so we will not go very far into this subject. Consider a current distribution localized within a sphere of radius $R$ of the origin. We seek an approximate form for the vector potential $\mathbf{A}(\mathbf{x})$ for $|\mathbf{x}| >> R$. In our integral (6.34) we expand to lowest order in $|\mathbf{x}'| << |\mathbf{x}|$

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{|\mathbf{x}|} + \frac{\mathbf{x} \cdot \mathbf{x}'}{|\mathbf{x}|^3} + \cdots$$

(6.48)

to find

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi} \left( \frac{1}{|\mathbf{x}|} \int d^3 \mathbf{x}' \mathbf{J}(\mathbf{x}') + \int d^3 \mathbf{x}' (\mathbf{x} \cdot \mathbf{x}') \mathbf{J}(\mathbf{x}') + \cdots \right),$$

(6.49)

where in the second term the dot product is $\mathbf{x} \cdot \mathbf{x}'$. We expect the first (monopole) contribution to vanish. To derive this as well as some other simplifications we will need, we note a lemma. For a localized vector field $\mathbf{V}$, and two scalar functions (all sufficiently well-behaved) we have

$$\int d^3 \mathbf{x} f(\mathbf{V} \cdot \nabla) g + g(\mathbf{V} \cdot \nabla) f + fg \nabla \cdot \mathbf{V} = \int d^3 \mathbf{x} \nabla \cdot (fg \mathbf{V}) = 0,$$

(6.50)

using the divergence theorem and the fact that $\mathbf{V}$ vanishes at large-$r$. If $\nabla \cdot \mathbf{V} = 0$ the last term is absent. Applying this to $\mathbf{J}$ with the choice $f = 1$, $g = x'_i$ we have

$$\int d^3 x' \mathbf{J}_i(\mathbf{x}') = 0,$$

(6.51)

validating our expectations. Continuing to the next order, we again use our lemma to simplify things. With $f = x'_i$, $g = x'_j$ (6.50) yields

$$\int d^3 x' (x'_i J_j + x'_j J_i) = 0,$$

(6.52)

so that we have

$$\int d^3 x' (\mathbf{x} \cdot \mathbf{x}') \mathbf{J}(\mathbf{x}') = \frac{1}{2} \int d^3 x' ((\mathbf{x} \cdot \mathbf{x}') \mathbf{J} - (\mathbf{x} \cdot \mathbf{J}) \mathbf{x}') = -\frac{1}{2} \mathbf{x} \times \int d^3 x' \mathbf{x}' \times \mathbf{J}.$$

(6.53)

Defining the magnetization, or dipole density

$$\mathcal{M}(\mathbf{x}) = \frac{1}{2} \mathbf{x} \times \mathbf{J}(\mathbf{x})$$

(6.54)
and its integral the *magnetic dipole moment*

\[
m = \int d^3x' M(x'),
\]  

(6.55)

we have at first nonvanishing order

\[
A(x) = \frac{\mu_0}{4\pi} \frac{m \times x}{|x|^3}.
\]  

(6.56)

In the simplifying case that the current is confined to a plane, as in our example, \( m \) is clearly perpendicular to the plane. For a planar loop of current-carrying wire we find

\[
m = \frac{1}{2} \oint x \times dl
\]  

(6.57)

and the magnitude of the dipole moment is the product of the current and the area of the wire loop regardless of its shape.

The magnetic field at large distances from any localized current distribution follows from (6.56) and we find

\[
B(x) = \frac{\mu_0}{4\pi} \nabla \times \left( m \times \frac{x}{|x|^3} \right).
\]  

(6.58)

Now we play vector identities

\[
\nabla \times \left( m \times \frac{x}{|x|^3} \right) = m \nabla \cdot \left( \frac{x}{|x|^3} \right) - (m \cdot \nabla) \left( \frac{x}{|x|^3} \right) = -m + 3\hat{r}(m \cdot \hat{r}) |x|^3
\]  

(6.59)

so that

\[
B(x) = \frac{\mu_0}{4\pi} \frac{-m + 3\hat{r}(m \cdot \hat{r})}{|x|^3}
\]  

(6.60)

is identical in form to the electric field of a dipole.

As in the electrostatic case, there is a contact term contribution that is generated in taking the limit of a pure dipole. Not surprisingly, since the limit has a very different nature, this contact term has a different coefficient. We compute, as we did, the integral of \( B \) over a ball, and find that results differ depending on whether or not the sphere contains the localized current distribution.

Indeed, we have, for a ball centered at the origin,

\[
\int_{r<R} d^3x B(x) = \int_{r<R} d^3x \nabla \times A(x) = R^2 \int_{r=R} d\Omega \hat{r} \times A,
\]  

(6.61)
after integrating by parts. Substituting from (6.34) for $A$ we have

$$\int_{r<R} d^3x \mathbf{B}(x) = -\frac{\mu_0}{4\pi} R^2 \int d^3x' \mathbf{J}(x') \times \int d\Omega \frac{\hat{r}}{|x - x'|}. \quad (6.62)$$

This last integral is the one I tried so hard to do in class. Let’s do it here. Selecting $x' = r'\hat{z}$ we see that

$$\int d\Omega \frac{\hat{r}}{|x - x'|} = 2\pi \int_{-1}^{1} d(cos \theta) \cos \theta \sum_{l \geq 0} \frac{r'^l}{r'^{l+1}} P_l(\cos \theta) \quad (6.63)$$

$$= \frac{2\pi r_< \hat{z}}{r_>^2} \int_{-1}^{1} d(cos \theta) \cos^2(\theta) = \frac{4\pi r_< r_<}{3} \frac{r_<}{r_>^2}.$$

where $r_<$ are the smaller (larger) among $R$ and $r'$ (both constant in the integral as $x$ moves around the sphere).

In all, we now have

$$\int_{r<R} d^3x \mathbf{B}(x) = \frac{\mu_0}{3} \int d^3x' \left( \frac{R^2 r_<}{r_>^2} \right) x' \times \mathbf{J}(x') \quad (6.64)$$

When the entire current distribution is within the sphere we may set $r_< = r'$ to find

$$\int_{r<R} d^3x \mathbf{B}(x) = \frac{2\mu_0}{3} \mathbf{m}, \quad (6.65)$$

while when the entire distribution lies outside the sphere, we have $r_< = R$ and

$$\int_{r<R} d^3x \mathbf{B}(x) = \frac{4\pi R^3}{3} \mathbf{B}(0) \quad (6.66)$$

The difference is due to a contact term; the field of a magnetic dipole is in fact given by

$$\mathbf{B}(x) = \frac{\mu_0}{4\pi} \left( \frac{3\hat{r}(\mathbf{m} \cdot \hat{r}) - \mathbf{m}}{|x|^3} + \frac{8\pi}{3} \mathbf{m}\delta^{(3)}(x) \right). \quad (6.67)$$

This is to be contrasted with the result for an electric dipole

$$\mathbf{E}(x) = k \left( \frac{3\hat{r}(\mathbf{p} \cdot \hat{r}) - \mathbf{p}}{|x|^3} - \frac{4\pi}{3} \mathbf{p}\delta^{(3)}(x) \right). \quad (6.68)$$

The difference suggests an interpretation of the contact term which becomes clear when we create a “pure dipole” as a limit of a charge or current distribution. If we construct an electric dipole from two charges $\pm q$ a distance $d$ apart in the limit $d \to 0$ with $qd$ constant, so that the “far field” approximation is valid everywhere, this neglects the large
fields encountered at the origin as we take the limit. This also occurs when we create a 
pure magnetic dipole as the limit of a loop with area \( A \to 0 \) carrying a current \( I \) with \( IA \) held constant in the limit. We then see that the fields at the origin are directed in opposite 
directions (relative to the far fields) in the two cases.

In (6.1) we defined a magnetic field as that which creates a torque on magnetic dipoles. 
Here we called a dipole the charge distribution creating a dipole field. It is gratifying to 
find that these two definitions agree. Indeed, consider a localized current distribution \( \mathbf{J}(x) \) 
in the presence of a slowly-varying magnetic field \( \mathbf{B}(x) \), and compute the force using (6.19). 
Expanding the field as 
\[
\mathbf{B}(x) = \mathbf{B}(0) + (x \cdot \nabla)\mathbf{B}(0) + \cdots,
\]
we find 
\[
\mathbf{F} = \int d^3x \mathbf{J}(x) \times [\mathbf{B}(0) + (x \cdot \nabla)\mathbf{B}(0) + \cdots].
\]
The first term vanishes as we showed above; for a steady state current distribution the 
integral of \( \mathbf{J} \) vanishes. In the second term we use the manipulations used above in (6.53) 
to find 
\[
\mathbf{F} = -\int d^3x (x \cdot \nabla)\mathbf{B}(0) \times \mathbf{J}(x) = \frac{1}{2} \left( \int d^3x x \times \mathbf{J}(x) \times \nabla \right) \mathbf{B}(0)
\]
\[
= (\mathbf{m} \times \nabla)\mathbf{B}(0) - \nabla (\mathbf{m} \cdot \mathbf{B})
\]
where we have in the last line expanded the triple product and \( \mathbf{m} \) is a constant in taking 
the gradient.

Similarly, expanding in (6.20) we find a contribution at leading order 
\[
\mathbf{N} = \int d^3x x \times (\mathbf{J} \times \mathbf{B}(0)) = \int d^3x [(\mathbf{x} \cdot \mathbf{B}(0))\mathbf{J}(x) - (\mathbf{x} \cdot \mathbf{J}(x))\mathbf{B}(0)].
\]
The second term vanishes upon integration using (6.52). The first term, using the by now 
familiar manipulation gives 
\[
\mathbf{N} = \mathbf{m} \times \mathbf{B}(0)
\]
to leading order. Comparing this to (6.1) we see that our definitions of magnetic dipoles 
are consistent.
6.7. Macroscopic Equations

As with electrostatics, microscopic properties of a medium can modify the equations obeyed by the observed macroscopic fields. Specifically, atoms and molecules exhibit magnetic dipoles (classically one could think of these as associated to the current describing the motions of charged particles within them; as we know this is misleading) and these respond to external fields. Upon everaging over microscopic fields, we observe an effective density of magnetization $\mathbf{M}(\mathbf{x})$ which enters our equations. Averaging over zero we do not get a contribution, so the homogeneous equation

$$ \nabla \times \mathbf{B} = 0 $$

is not modified and we may continue to use a vector potential. In Coulomb gauge, we may take this into account by adding to the vector potential induced by external current distributions the contribution

$$ \mathbf{A}_M(\mathbf{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\mathbf{M}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} . \quad (6.74) $$

We can rewrite this term as

$$ \mathbf{A}_M(\mathbf{x}) = \frac{\mu_0}{4\pi} \int_V d^3x' \mathbf{M}(\mathbf{x}') \times \nabla' \left( \frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = \frac{\mu_0}{4\pi} \int_V d^3x' \frac{\nabla' \times \mathbf{M}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \int_{\partial V} \frac{\mathbf{M}(\mathbf{x}') \times d\mathbf{a}}{|\mathbf{x} - \mathbf{x}'|} , \quad (6.75) $$

where the boundary term will be absent if $\mathbf{M}$ is localized and the surface is taken outside its support. The magnetization thus contributes an effective current density

$$ \mathbf{J}_M = \nabla \times \mathbf{M} \quad (6.76) $$

and, on the boundary of a magnetic material, an effective surface current

$$ \mathbf{K}_M = \mathbf{M} \times \hat{\mathbf{n}} . \quad (6.77) $$

The magnetization thus modifies the equation satisfied by the macroscopic field to

$$ \nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \nabla \times \mathbf{M}) , \quad (6.78) $$

suggesting that we define a new macroscopic field

$$ \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \quad (6.79) $$
which satisfies an equation
\[ \nabla \times \mathbf{H} = \mathbf{J} \quad (6.80) \]
containing only the external current. Conventionally, \( \mathbf{H} \) is referred to as the *magnetic field* while \( \mathbf{B} \) is the *magnetic induction*. Calling \( \mathbf{B} \) the magnetic field, as we have done to this point, is imprecise, but it is an imprecision that is quite common among microscopic physicists, who work in regimes where \( \mathbf{H} \) is rarely needed or introduced, and when no confusion can arise we will continue to use it.

To (6.80) and (6.23) we need of course to add a *constitutive relation* depending on the medium and expressing the response to applied fields in the form of a relation between \( \mathbf{H} \) and \( \mathbf{B} \). We write this as
\[ \mathbf{B} = \mu \mathbf{H} , \quad (6.81) \]
with \( \mu \) a parameter depending on the material and called the *magnetic permeability*. We will return to this momentarily.

In solving magnetostatic problems in which interfaces between materials with different magnetic properties occur, we will use the boundary conditions on the fields. Thus at a boundary with normal \( \mathbf{n} \) pointing from region 1 to region 2
\[ \begin{align*}
\nabla \cdot \mathbf{B} &= 0 \\
\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) &= 0 \\
\nabla \times \mathbf{H} &= \mathbf{J} \\
\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) &= \mathbf{K} ,
\end{align*} \quad (6.82) \]
where \( \mathbf{K} \) is an applied (external) surface current density.

As in the electrostatic case, the magnetic response to an applied field is measured by the permeability or be the related *magnetic susceptibility* \( \chi_m \) defined by
\[ \mathbf{M}(\mathbf{x}) = \chi_m \mu_0 \mathbf{H}(\mathbf{x}). \quad (6.83) \]
Comparing this to (5.8) we note that the analogy with electric susceptibility is imperfect, the latter being defined by the relation between the polarization \( \mathbf{P} \) and the applied electric field \( \mathbf{E} \). The difference is perhaps historical in origin, but not without basis in physics, as we will see. The susceptibility and permeability are related by
\[ \mu = \mu_0 (1 + \chi_m) . \quad (6.84) \]
The relation (6.81) is of course the simplest possible, and more interesting situations exist. These are of special importance in the magnetic case both because they are more
common and because historically the first known sources of magnetic field were magnetic materials and not macroscopic current distributions. In particular, in anisotropic media the permeability becomes a tensor and the magnetization, field and induction are not all parallel. Moreover, the relation between \( \mathbf{M} \) and \( \mathbf{H} \) can be nonlinear (expressed as a dependence of \( \mu \) on \( \mathbf{H} \)), or even not single-valued so that defining a permeability becomes impossible.

The case analogous to a dielectric is that of a diamagnetic material. Typically these are materials in which the applied field induces molecular dipole moments. These will be oriented opposite to the field. This leads to a negative susceptibility and a permeability \( \mu < \mu_0 \). This seems counter to the terminology in electrostatics, where dielectrics had \( \epsilon > \epsilon_0 \); the difference arises from the different definitions of these, compare (5.9) to (6.81). Like a dielectric, a diamagnetic material will be attracted to regions of strong magnetic fields. Typically, \( \mu \) differs from \( \mu_0 \) by less than a few parts in \( 10^5 \). In paramagnetic materials, molecules have intrinsic magnetic moments, which will align with an applied field. This leads to a positive susceptibility and \( \mu > \mu_0 \). The fact that these two effects have the opposite sign is a definite departure from the behavior of electric dipole moments, induced and intrinsic.

An extreme and important case is that of ferromagnetic materials. In these materials, interactions between the atoms or molecules tend to align their magnetic moments. This means that the response to an applied field can be extreme, leading to a magnetization far larger than the applied \( \mathbf{H} \) produced by free currents. Weak-field values of \( \mu/\mu_0 \) can be as high as \( 10^6 \). As the applied field is increased, the magnetization increases until saturation sets in, and a linear increase of \( \mathbf{B} \) with \( \mathbf{H} \) obtains.

As an example of the fields in the presence of magnetization, as well as of the boundary conditions, we consider the case of a sphere of hard ferromagnetic material, with a uniform magnetization \( \mathbf{M} = m \hat{z} \). Since there are no free currents in our problem, the magnetic field satisfies \( \nabla \times \mathbf{H} = 0 \) and we can write

\[
\mathbf{H} = -\nabla \Phi_M ,
\]

where the equation satisfied by \( \Phi_M \) is

\[
\nabla \cdot \mathbf{B} = \mu_0 \nabla \cdot (\mathbf{H} + \mathbf{M}) = 0 ,
\]

or

\[
\nabla^2 \Phi_M = \nabla \cdot \mathbf{M} .
\]
If there are no boundaries, we then have
\[
\Phi_M(x) = -\frac{1}{4\pi} \int d^3 x' \frac{\nabla' \cdot M}{|x - x'|}.
\] (6.88)

The divergence of \( M \) acts as a source of “magnetic charge” \( \rho_M = \nabla \cdot M \) and if there are discontinuities in the magnetization, as at the boundary between magnetic and non-magnetic materials, we can treat this as an effective magnetic surface-charge density
\[
\sigma_M = \hat{n}_{12} \cdot (M_1 - M_2).
\] (6.89)

These magnetic charges are of course fictitious; they are a useful tool for solving problems with permanent magnets, for which \( M \) is known. They lead to a calculation of \( H \) using techniques from electrostatics; in fact, \( H \) itself, absent external currents to generate it and a constitutive equation to relate it to \( B \), is a physically meaningless, if computationally convenient, construction in these cases.

In our case, we have a constant magnetization inside the magnet, so that the potential is determined by a surface charge density given by \( \sigma_M = M \cos \theta \) and we have immediately
\[
\Phi_M(x) = \frac{Ma^2}{4\pi} \int d\Omega' \frac{\cos \theta'}{|x - x'|}.
\] (6.90)

Inserting the expansion in Legendre polynomials, we see that only the \( l = 1 \) term survives the integration and
\[
\Phi_M(x) = \frac{M a^2}{3} \frac{r <}{r^2} \cos \theta.
\] (6.91)

Inside the magnet, \( r < a \) and we have
\[
\Phi_M = \frac{M z}{3} \quad H = -\frac{1}{3} M \quad B = \frac{2\mu_0}{3} M.
\] (6.92)

Thus \( B \) is parallel to \( M \) but \( H \) antiparallel. Outside the sphere \( r > a \) and
\[
\Phi_M = \frac{Ma^3 \cos \theta}{3 r^2}
\] (6.93)
is the potential of a dipole with moment
\[
m = \frac{4\pi a^3}{3} M
\] (6.94)
as expected, so that \( B = -\mu_0 \nabla \Phi_M \) is a dipole field. The special symmetry of a sphere means this is not only the asymptotic form of the field but an exact expression.
By taking $\mathbf{M}$ to be independent of external fields, we can study the case of a magnetized sphere in an external field by simply superposing the external fields on those found above. The fields we find inside the sphere,

\[
\mathbf{B}_{\text{in}} = \mathbf{B}_0 + \frac{2\mu_0}{3} \mathbf{M}
\]
\[
\mathbf{H}_{\text{in}} = \frac{1}{\mu_0} \mathbf{B}_0 - \frac{1}{3} \mathbf{M}
\]

satisfy, if we take $\mathbf{B}_0$ parallel (or antiparallel) to $\mathbf{M}$

\[
\mathbf{H}_{\text{in}} = \mu \mathbf{B}_{\text{in}}
\]

with

\[
\mu/\mu_0 = \frac{B_0 + \frac{2\mu_0}{3} M}{B_0 - \frac{2\mu_0}{3} M}.
\]

This is a neat trick. What we have found is in fact that a uniform sphere of magnetic permeability $\mu$ placed in a uniform external field will develop a uniform magnetization such that (6.97) is satisfied, i.e.

\[
\mathbf{M} = \frac{3}{\mu_0} \left( \frac{\mu - \mu_0}{\mu + 2\mu_0} \right) \mathbf{B}_0.
\]

### 6.8. Charged Particles and Magnetic Fields

In discussing the interactions of electric currents with magnetic fields, we have used continuous, static current distributions. Of course, this is an idealization. In the real world, currents are carried by moving charged particles, and the interactions we have found reflect the microscopic interactions of the fields with charged particles. It is worthwhile to pause for a moment to think about this aspect of the physics.

As a first attempt, we can start with our expression (6.19) for the force on a current-carrying wire. If we consider the current density as reflecting the motion of particles of charge $q$ we have

\[
\mathbf{J} = n q \mathbf{v},
\]

where $\mathbf{v}$ is the (average, should that be relevant) velocity of the charged particles, and $n$ their number density. (6.19) looks like a local interaction, and could be reproduced by a force law

\[
\mathbf{F}_{\text{mag}} = q \mathbf{v} \times \mathbf{B}.
\]
Adding the familiar electric force, we have the *Lorentz force*

\[ \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) . \]  

(6.101)

This is a bit puzzling, because the force depends upon the velocity of the particle, and while forces dependent on relative velocities are commonplace, a dependence on absolute velocity would violate Galilean invariance. To recover this important property, we could construct transformation properties of the fields, such that (6.101) holds in any frame. This turns out to be tricky; with hindsight the reason is clear: Galilean invariance is an approximate symmetry of nature, valid for small velocities. The magnetic contribution to (6.101) turns out to be of the order of the relativistic corrections. In fact, somewhat fortuitously, this form of the force law turns out to be both relativistically covariant and correct with no need for relativistic corrections.

We will have a chance to discuss all of this in far more detail, for now we harvest the simple results that can be drawn from what we have. The first observation is that, since they act in a direction perpendicular to the velocity, purely magnetic forces do no work. The motion of a particle in a constant magnetic field is helical. Motion in a direction parallel to the field is free, while in the plane perpendicular to the field we have circular motion with radius given by

\[ \frac{mv_\perp^2}{R} = qv_\perp B , \]  

(6.102)
yielding

\[ R = \frac{v_\perp}{B} \left( \frac{m}{q} \right) , \]  

(6.103)

allowing the famous measurement of \( e/m \) for the electron, by J.J. Thomson (this equation does receive relativistic corrections at high velocities). The angular velocity is

\[ \omega = \frac{v}{R} = \frac{qB}{m} \]  

(6.104)

and is independent of both \( R \) and \( v_\perp \).

When the magnetic field is not constant, motion along the field lines is not free but for slowly-varying fields we can treat the circular motion as a current loop and use our equation (6.71) for the force on such a current. This shows that a charged particle in a slowly varying field will be attracted to regions of *weak* field, and repelled from regions of strong fields. This forms the basis of magnetic “mirrors” and “bottles” for confining
plasma, and in another context for the existence of captive charged particles in radiation belts trapped by the Earth’s magnetic field and bouncing between the polar regions.

Emboldened by this success, we can try to push the relation between electric current and particle motion further. In our case above, with the current carried by charged particles of uniform charge $q$, the number density $n$ determines the charge density

$$\rho = qn , \quad (6.105)$$

as well as the mass density of the charged particles, so that the electric current is just $q/m$ times the momentum density. In particular, this means that the magnetic dipole moment given by (6.55) and (6.54) is related to the angular momentum of the motion via

$$\mathbf{m} = \frac{1}{2} \int d^3x \times \mathbf{J}(x) = \frac{q}{2m} \int d^3x \times (nm\mathbf{v}) = \frac{q}{2m} \mathbf{L} . \quad (6.106)$$

The ratio between these two (pseudo-)vector properties is called the gyromagnetic ratio. This relation, indeed, holds for orbital motions, even those of electrons on an atomic scale. For the intrinsic angular momentum (spin) and magnetic dipole moment of an electron, however, it fails. In general, we define the $g$-factor as the ratio of the gyromagnetic factor to $q/2m$. For an electron, a relativistic calculation at lowest order yields $g = 2$. In fact, this is corrected by higher-order effects and, measured to some 12 significant digits, is one of the most carefully measured (and calculated) properties of the natural world.

One can try to push things too far. It is tempting, and was very natural in nineteenth-century physics, to continue our identification of localized currents as moving charged particles to the generation of the fields (recall our triumphant discovery that what creates a dipole field interacts with external fields as a dipole). Inserting (6.99) into the Biot-Savart law (6.21) we would find for a moving charged particle

$$\mathbf{B} = \frac{\mu_0 q'v \times \mathbf{x}}{4\pi r^3} . \quad (6.107)$$

for the field a distance $r$ from a moving particle. Of course, this is unfair because we certainly cannot consider one moving particle a static current and indeed we have time-dependent fields (electric as well as magnetic) in this situation. Indeed, if we try to put our successful Lorentz force formula together with (6.107) we find

$$\mathbf{F} = \frac{kqq'({x - x'}^3)}{|x - x'|^3} + \frac{\mu_0 q'v \times (v' \times (x - x'))}{4\pi |x - x'|^3} . \quad (6.108)$$
This is now really terrible, because the force between two charges depends on their absolute velocities. Of course, again the magnetic force, of order $v^2$, is comparable to relativistic corrections, and we will find a corrected version of this that works by including the electromagnetic field as a dynamic entity. But for nineteenth century physics this was a real problem, and modifications of this equation proposed by many prominent physicists did not lead to a satisfactory resolution. As a way out, Maxwell and others were moved to suggest the existence of an absolute frame, the “ether,” with respect to which the velocities are to be measured, and to attempt to detect its presence. The failure foreshadows the special theory of relativity, on which more soon.