1. Time dilation is a property of spacetime, hence universal. Any moving clock constructed from physical objects governed by relativistically invariant laws will in fact be observed to slow when in motion. A famous example is a light clock. This is a device containing two mirrors separated by 0.5 m. and facing each other. A beam of light bounces between the two, triggering a counter on one of the mirrors each time it encounters it, so that the counter tells time in light-meters.

(a) Consider a light clock moving with velocity $v$ perpendicular to its axis (the normal to the mirrors) relative to a collection of synchronized inertial observers $S$. Find explicitly the time, as measured by these observers, between successive counts of the light clock, using the fact that light moves with unit velocity in any frame, and show that this agrees with our results for time dilation. To do this you must describe the trajectory of the bouncing light beam as seen by the observers in $S$.

(b) Repeat the calculation for the case when the clock is rotated so that its axis is aligned with $v$. Remember length contraction here.

2.

(a) Two inertial frames $S$ and $S'$ are in standard configuration, and $S'$ has velocity $v$ relative to $S$. An observer in $S$ sees a stick of length $b$ fixed at an angle $\theta$ from the $x$-axis. What is the length and orientation of the stick as measured by an observer in $S'$?

(b) A meter stick lies along the $x$-axis and approaches the origin, moving along its length, with velocity $v_x$. A very thin plate, parallel to the $xz$ plane in the laboratory, moves upward in the $y$ direction with velocity $v_y$. The plate has a circular hole with diameter 1m centered on the $y$-axis. The center of the meter stick arrives at the origin at the same time as the center of the hole in the plate. In the laboratory frame, the meter stick is Lorentz contracted, so fits in the hole in the plate, and plate and stick continue on their paths without a collision. In the rest frame of the meter stick, however, it is the plate, and the hole in it, that is
contracted along the $x$-axis. This would seem to predict a collision, contradicting the requirement that physical predictions must be invariant. Use a careful analysis to show which of the two predictions is in fact correct.

3. The *proper acceleration* of an object is defined as its acceleration in its instantaneous rest frame, i.e. the acceleration measured by inertial observers relative to whom the object’s instantaneous velocity vanishes.

(a) Find the position at time $t$ (in an inertial frame $S$) of a rocket ship that starts from rest in $S$ and moves along the $x$ axis with constant proper acceleration $\alpha$.

(b) At time $t_0$, a radio signal is sent to the rocket from the spaceport whence it departed. Find the time $T$ at which the signal is received. Show that there is a maximal value of $t_0$ after which the ship is out of contact with the spaceport.

(c) For $t_0$ such that the signal is received by the ship, find the frequency $\nu$ at which the signal is observed if it is emitted at frequency $\nu_0$.

(d) Find the proper acceleration of a particle moving at uniform speed $u$ around a circle of radius $R$.

4. Assume that a rocket ship leaves the earth in the year 2100. One of a set of twins born in 2080 remains on earth; the other rides in the rocket. The rocket ship is so constructed that is has an acceleration $g$ in its own rest frame (this makes the occupants feel at home). It accelerates in a straight-line path for 5 years (by its own clock), decelerates at the same rate for 5 more years, turns around, accelerates for 5 years, decelerates for 5 years, and lands on earth. The twin in the rocket is 40 years old.

(a) What year is it on earth when the ship lands?

(b) How far away from the earth did the rocket ship travel?

5. An infinitesimal Lorentz transformation and its inverse can be written

$$x'^\mu = (\eta^{\mu\nu} + \epsilon^{\mu\nu})x_\nu,$$

$$x^\mu = (\eta^{\mu\nu} + \epsilon'^{\mu\nu})x'_\nu,$$

where $\epsilon^{\mu\nu}$ and $\epsilon'^{\mu\nu}$ are infinitesimal.

(a) Show from the definition of the inverse that $\epsilon'^{\mu\nu} = -\epsilon^{\mu\nu}$. 

(b) Show from the preservation of the invariant inner product that $\epsilon^{\mu\nu} = -\epsilon^{\nu\mu}$.

(c) Use this to count the number of parameters needed to determine an infinitesimal Lorentz transformation. This is the dimension of the Lorentz group. Can you identify the parameters?

6. Here is yet another version of the twin paradox. Consider a spacetime in which one of the spatial directions, say the $x$-direction, is a circle of circumference $L$, i.e. $x$ and $x + L$ are the same point in space. An observer, $A$, at rest is passed by another observer, $B$, moving at relative velocity $v$ in the $x$-direction, and the two synchronize clocks as they pass. As usual, each thinks the other’s clock is running slow.

(a) Unlike usual, they meet up again at time $T = L/v$, neither having undergone any acceleration, and compare clocks. Whose clock shows the greater elapsed time? By what factor?

(b) This seems paradoxical, since locally there is nothing that breaks the symmetry between the two observers. Describe an experiment that either observer could do alone to determine whether his/her clock would register the greater time in the experiment of (a). (Hint: the distinguishing feature of this version is the non-trivial global topology of spacetime, which locally looks a lot like Minkowski space, so such an experiment should be sensitive to this feature.)