1. In class we found the expression for the Christoffel connection

\[ \Gamma^\mu_{\lambda\nu} = \frac{1}{2} g^{\mu\sigma} \left( \partial_\lambda g_{\sigma\nu} + \partial_\nu g_{\lambda\sigma} - \partial_\sigma g_{\lambda\nu} \right). \]

We showed that this is metric compatible, and it is certainly torsion-free. Show that this is a connection by explicitly computing how it transforms under a change of coordinates.

2. Let \( d\Omega = d\theta^2 + \sin^2 \theta d\phi^2 \) denote the line element on a 2-sphere of unit radius.

(a) Find the Christoffel components on the sphere in these coordinates. Solve the geodesic equation with initial condition \( \theta = \pi/2 \) and \( \dot{\theta} = 0 \) and use this to find the length of the equator.

(b) The metric on flat 3-dimensional Euclidean space is \( ds^2 = dx^2 + dy^2 + dz^2 \). In problem 4 you found the form this takes in polar coordinates \((r, \theta, \phi)\) Calculate the Christoffel components \( \Gamma^i_{jk} \) in this coordinate system, write down the components of the geodesic equation, and show that geodesics through the origin correspond to straight lines in Cartesian coordinates.

3. Covariant integration by parts is the statement that

\[ \int d^n x \sqrt{|g|} \left( \nabla_\lambda S^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} \right) T_{\nu_1 \ldots \nu_l}^{\nu_1 \ldots \nu_l} = - \int d^n x \sqrt{|g|} S^{\mu_1 \ldots \mu_k}_{\nu_1 \ldots \nu_l} \left( \nabla_\lambda T^{\lambda \nu_1 \ldots \nu_l}_{\mu_1 \ldots \mu_k} \right) \]

for arbitrary tensors \( S, T \) that vanish on an asymptotic boundary.

(a) Show that \( \Gamma^\mu_{\nu\mu} = \frac{1}{\sqrt{-|g|}} \partial_\nu \sqrt{-|g|} \). This is an extremely useful identity.

Hint: Use the expression for the inverse of a matrix in terms of its determinant and its minors, which can be written as

\[ g^{\mu\nu} = \frac{1}{3! |g|} \epsilon^{\mu_1 \mu_2 \mu_3} \epsilon_{\nu_1 \nu_2 \nu_3} \nu \ g_{\mu_1 \nu_1} g_{\mu_2 \nu_2} g_{\mu_3 \nu_3} \]
where $\epsilon$ is the antisymmetric symbol we defined with values 0, ±1 (so not a tensor) and

$$|g| = \frac{1}{4!} \epsilon^{\mu_1\mu_2\mu_3\mu_4} \epsilon_{\nu_1\nu_2\nu_3\nu_4} \epsilon_{\mu_1\nu_1\nu_2\nu_3\nu_4\nu_4} \epsilon_{\nu_1\nu_2\nu_3\nu_4} g_{\mu_1\nu_1} g_{\mu_2\nu_2} g_{\mu_3\nu_3} g_{\mu_4\nu_4} .$$

(b) Use the results of (a) to prove the above formula

Hint: Evaluate $\partial_{\nu} \left( \sqrt{|g|} S^{\cdots T \cdots} \right)$ and use Stokes’s theorem.

4. Consider a spacetime with metric

$$ds^2 = dt^2 - dx^2$$

where

$$dx^2 = g_{ij}(t, x) dx^i dx^j$$

is an arbitrary Riemannian metric on what we may call space.

(a) Show that $x = \text{const}$ is a geodesic.

(b) If $g_{ij}$ is $t$-independent, show that every geodesic follows a geodesic track through space.

5. In this problem we will use our newfangled technology to solve an old familiar problem in the hope that this will help us understand the meaning of some of the things we are playing with. The problem is nonrelativistic projectile motion in a uniform gravitational field. The solution should be $d^2z/dt^2 = -g$ with all other components of acceleration vanishing. We want to find the metric and Christoffel connection describing this background. To get these we start by constructing inertial (freely falling) coordinates. In this simplified problem we can in fact find $\text{globally}$ inertial coordinates, because what we are in fact describing is not real gravity but the fake gravity observed in an accelerating frame. The inertial coordinates are determined up to rotation and boost, but a simple choice is just $z' = z + gt^2/2$ with all other $x'^\mu = x^\mu$. In the inertial coordinates, of course, $g'_{\mu\nu} = \eta_{\mu\nu}$ and Newton’s first law holds.

(a) Using the coordinate transformation properties of the metric, find $g_{\mu\nu}$ and $g^{\mu\nu}$ in the noninertial coordinates. To simplify your work you can note that the problem is essentially two-dimensional, anything to do with $x, y$ is trivial.
(b) Check your results in (a) by verifying that \( g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda \) as it must be. Restricting to the spatial components of the metric \( g_{ij} \) gives a description of a time-slice, or what we normally call space. What does the form of our metric tell you about the shape of space in this problem? Does that make sense?

(c) Now, using the metric components you computed above, find the components \( \Gamma^\mu_{\nu\lambda} \) of the Christoffel connection in the noninertial frame.

(d) Reobtain the answer to (c) by using the non-tensorial transformation properties of \( \Gamma \) together with the fact that in the (global) inertial frame \( \Gamma' = 0 \) to find \( \Gamma \) directly.

(e) Now, finally, write down the equations for geodesics in the noninertial coordinates

\[
\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0
\]

and show that they lead to the expected result for nonrelativistic projectile motion.

6. Consider the two-sphere with the metric \( d\Omega = d\theta^2 + \sin^2 \theta d\phi^2 \). You found the Christoffel connection for this in problem 2. Use this to perform the following exercise in parallel transport. We start with the tangent vector \( u \) (of unit length) to the curve \( \phi = 0 \) (a line of longitude) at the North pole \( \theta = 0 \). This is then parallel transported, i.e. moved along the curves listed below so that

\[
\frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{ds} u^\lambda = 0 ,
\]

where \( s \) is any parameter (note that this equation is in fact invariant under reparameterizations of the curve) along the curve and we use the initial condition to solve this finding a tangent vector \( u \) at any point along the curve. In our case, we transport \( u \) along these curves:

(a) Along the curve \( \phi = 0 \) from \( \theta = 0 \) to \( \theta = \theta_0 \).

(b) Along the curve \( \theta = 0 \) from \( \phi = 0 \) to \( \phi = \phi_0 \).

(c) Along the curve \( \phi = \phi_0 \) from \( \theta = \theta_0 \) back to the North pole \( \theta = 0 \).

Find the transported vector at the end point of each of these stages. Note that the final stage ends at the starting point, but we did not find the same vector we started
with. Parallel transport along a closed curve can be nontrivial. By what angle has our initial vector been rotated when we bring it back to the pole? You may find it helpful to augment your calculations here by drawing the sphere and the transported tangent vectors.