1. In class we found the expression for the Christoffel connection
\[ \Gamma_{\lambda\nu}^{\mu} = \frac{1}{2} g^{\mu\sigma} \left( \partial_{\lambda} g_{\sigma\nu} + \partial_{\nu} g_{\lambda\sigma} - \partial_{\sigma} g_{\lambda\nu} \right). \]

We showed that this is metric compatible, and it is certainly torsion-free. Show that this is a connection by explicitly computing how it transforms under a change of coordinates.

We need to show
\[ \Gamma'_{\lambda\nu}^{\mu} = \frac{1}{2} g'^{\mu\sigma} \left( \partial'_{\lambda} g'_{\sigma\nu} + \partial'_{\nu} g'_{\lambda\sigma} - \partial'_{\sigma} g'_{\lambda\nu} \right) = \frac{\partial x^\rho}{\partial x'^\lambda} \frac{\partial x^\sigma}{\partial x'^\nu} \Gamma_{\rho\sigma} + \frac{1}{2} g'^{\mu\sigma} \left[ \partial'_{\nu} \left( \frac{\partial x^\alpha}{\partial x'^\sigma} \frac{\partial x^\beta}{\partial x'^\lambda} \right) g_{\alpha\beta} + \partial'_{\lambda} \left( \frac{\partial x^\alpha}{\partial x'^\sigma} \frac{\partial x^\beta}{\partial x'^\nu} \right) g_{\alpha\beta} - \partial'_{\sigma} \left( \frac{\partial x^\alpha}{\partial x'^\nu} \frac{\partial x^\beta}{\partial x'^\lambda} \right) g_{\alpha\beta} \right]. \]

The first line is fine, let’s work on the second line. Applying the product rule to expand the derivatives we see that four of the resulting terms cancel pairwise, while the other two are related by exchanging \( \alpha \) with \( \beta \) and thus contribute the same due to the symmetry of \( g_{\alpha\beta} \). The result is that the second line is
\[ g'^{\mu\sigma} g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\sigma} \frac{\partial^2 x^\beta}{\partial x'^\lambda \partial x'^\nu}. \]

We now use the transformation property of \( g \) to find
\[ \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\sigma}{\partial x^\tau} g'^{\rho\sigma} g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\sigma} \frac{\partial^2 x^\beta}{\partial x'^\lambda \partial x'^\nu} = \delta^\tau_{\lambda} \frac{\partial^2 x^\beta}{\partial x'^\tau \partial x'^\mu} = \frac{\partial x'^\mu}{\partial x^\rho} g'^{\rho\alpha} g_{\alpha\beta} \frac{\partial^2 x^\beta}{\partial x'^\lambda \partial x'^\nu} = \frac{\partial^2 x^\beta}{\partial x'^\lambda \partial x'^\nu}. \]
To find the form we prefer, apply an identity. Starting from
\[ \frac{\partial x^{\mu}}{\partial x^\rho} \frac{\partial x^\rho}{\partial x^\lambda} = \delta^{\mu}_\lambda \]
we apply the derivative \( \partial'_\nu = (\partial x^\sigma / \partial x'^\nu) \partial_\sigma \) to obtain
\[ \frac{\partial x^{\mu}}{\partial x^\rho} \frac{\partial^2 x^\rho}{\partial x^\lambda \partial x'^\nu} + \frac{\partial^2 x'^\mu}{\partial x^\rho \partial x^\sigma} \frac{\partial x^\rho}{\partial x^\lambda} \partial_\sigma = 0. \]
Inserting this we find in all
\[ \Gamma^{\mu}_{\lambda \nu} = \frac{\partial x^\rho}{\partial x'^\lambda} \frac{\partial x^\sigma}{\partial x'^\nu} \frac{\partial x^\mu}{\partial x^\tau} \Gamma^\tau_{\rho \sigma} - \frac{\partial x^\sigma}{\partial x'^\nu} \frac{\partial x^\rho}{\partial x^\tau} \frac{\partial^2 x^\mu}{\partial x^\lambda \partial x^\tau}. \]

2. Let \( d\Omega = d\theta^2 + \sin^2 \theta d\phi^2 \) denote the line element on a 2-sphere of unit radius.

(a) Find the Christoffel components on the sphere in these coordinates. Solve the geodesic equation with initial condition \( \theta = \pi/2 \) and \( \dot{\theta} = 0 \) and use this to find the length of the equator.

The nonzero Christoffel coefficients are, by direct computation
\[ \Gamma^{\theta}_{\phi \phi} = -\sin \theta \cos \theta \quad \Gamma^{\phi}_{\theta \phi} = \cot \theta. \]
The geodesic equations are thus
\[ \frac{dv^\theta}{ds} - \sin \theta \cos \theta \left( v^\phi \right)^2 = 0 \]
\[ \frac{dv^\phi}{ds} + 2 \cot \theta v^\theta v^\phi = 0. \]
With initial conditions \( v^\theta = 0 \) and \( \cos \theta = 0 \) we have a solution with \( v^\theta = 0 \) and \( v^\phi \) constant. The geodesic we find is the equator. To find its length we need to normalize \( v^2 = 1 \) which makes the parameter \( s \) proper distance along the geodesic. We find \( v^2 = \sin^2 \theta (v^\phi)^2 \) so at the equator we set \( v^\phi = 1 \). This means \( s = \phi \) (up to a constant) giving the equator length \( 2\pi \).

(b) The metric on flat 3-dimensional Euclidean space is \( ds^2 = dx^2 + dy^2 + dz^2 \). In problem 4 last week you found the form this takes in polar coordinates \( (r, \theta, \phi) \)
Calculate the Christoffel components \( \Gamma^i_{jk} \) in this coordinate system, write down
the components of the geodesic equation, and show that geodesics through the origin correspond to straight lines in Cartesian coordinates.

The nonvanishing components here are

\[
\begin{align*}
\Gamma^{r}_{\theta \theta} &= -r \\
\Gamma^{r}_{\phi \phi} &= -r \sin^2 \theta \\
\Gamma^{\theta}_{r \theta} &= r^{-1} \\
\Gamma^{\phi}_{r \phi} &= \cot \theta
\end{align*}
\]

The geodesic equations are thus

\[
\begin{align*}
\frac{d^2 v^r}{ds^2} - r (v^\theta)^2 - r \sin^2 \theta (v^\phi)^2 &= 0 \\
\frac{d^2 v^\theta}{ds^2} + 2r^{-1} v^r v^\theta - \cos \theta \sin \theta (v^\phi)^2 &= 0 \\
\frac{d^2 v^\phi}{ds^2} + 2r^{-1} v^r v^\phi + 2 \cot \theta v^\theta v^\phi &= 0
\end{align*}
\]

Straight lines through the origin are obtained by starting at an arbitrary point and setting as initial conditions \( v^\phi = v^\theta = 0 \). Clearly \( v^r \) constant as the only nonzero component of \( v \) is a solution.

3. Covariant integration by parts is the statement that

\[
\int d^n x \sqrt{|g|} \left( \nabla_\lambda S^\mu_{\nu_{1} \cdots \nu_{k}} \right) T^\lambda_{\mu_{1} \cdots \mu_{k}} = -\int d^n x \sqrt{|g|} S^\mu_{\nu_{1} \cdots \nu_{k}} \left( \nabla_\lambda T^\lambda_{\mu_{1} \cdots \mu_{k}} \right)
\]

for arbitrary tensors \( S, T \) that vanish on an asymptotic boundary.

(a) Show that \( \Gamma^\mu_{\nu \mu} = \frac{1}{\sqrt{-|g|}} \partial_\nu \sqrt{-|g|} \). This is an extremely useful identity.

Hint: Use the expression for the inverse of a matrix in terms of its determinant and its minors, which can be written as

\[
g^{\mu \nu} = \frac{1}{3!|g|} \epsilon^{\mu \nu_{1} \nu_{2} \nu_{3}} \epsilon_{\nu_{1} \nu_{2} \nu_{3} \nu} g_{\mu_{1}} g_{\mu_{2}} g_{\mu_{3}} g_{\nu_{4}}
\]

where \( \epsilon \) is the antisymmetric symbol we defined with values 0, \pm 1 (so not a tensor) and

\[
|g| = \frac{1}{4!} \epsilon^{\mu_{1} \mu_{2} \mu_{3} \mu_{4}} \epsilon_{\nu_{1} \nu_{2} \nu_{3} \nu_{4}} g_{\mu_{1}} g_{\mu_{2}} g_{\mu_{3}} g_{\nu_{4}} .
\]
First let’s note that
\[ \Gamma^\mu_{\nu\mu} = \frac{1}{2} g^{\mu\sigma} (\partial_\nu g_{\sigma\mu} + \partial_\mu g_{\nu\sigma} - \partial_\sigma g_{\mu\nu}) = \frac{1}{2} g^{\mu\sigma} \partial_\nu g_{\sigma\mu} \]
because the last two terms cancel using the symmetry of \( g^{\mu\sigma} \).

Now we compute using the hint
\[
\frac{1}{2} g^{\mu\sigma} \partial_\nu g_{\sigma\mu} = \frac{1}{2} \frac{1}{3!|g|} \epsilon^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} g_{\mu_1\nu_1} g_{\mu_2\nu_2} g_{\mu_3\nu_3} \partial_\nu g_{\mu\sigma} \\
= \frac{1}{2} \frac{1}{4!|g|} \epsilon^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} \partial_\nu (g_{\mu_1\nu_1} g_{\mu_2\nu_2} g_{\mu_3\nu_3} g_{\mu\sigma}) \\
= \frac{1}{2|g|} \partial_\nu (|g|) \\
= \frac{1}{\sqrt{-|g|}} \partial_\nu \sqrt{-|g|},
\]
where the second equality used the symmetry of \( \epsilon^2 \) under the simultaneous exchange of \( \mu, \sigma \) with \( \mu_i, \nu_i \) for any \( i \) and the Leibnitz rule.

(b) Use the results of (a) to prove the above formula

Hint: Evaluate \( \partial_\nu \left( \sqrt{|g|} S^{\cdot \cdot \cdot} T^{\nu \cdot \cdot \cdot} \right) \) and use Stokes’s theorem.

Let’s take the hint. Using the result of (a) we have
\[
\partial_\nu \left( \sqrt{-|g|} S^{\cdot \cdot \cdot} T^{\nu \cdot \cdot \cdot} \right) = \sqrt{-|g|} \partial_\nu (S^{\cdot \cdot \cdot} T^{\nu \cdot \cdot \cdot}) + \partial_\nu \left( \sqrt{-|g|} \right) S^{\cdot \cdot \cdot} T^{\nu \cdot \cdot \cdot} \\
= \sqrt{-|g|} (\partial_\nu + \Gamma^\lambda_{\nu\lambda}) S^{\cdot \cdot \cdot} T^{\nu \cdot \cdot \cdot} \\
= \sqrt{-|g|} D_\nu (S^{\cdot \cdot \cdot} T^{\nu \cdot \cdot \cdot}).
\]

Stokes’s theorem (or the divergence theorem, if you prefer to call it that) implies that for fields vanishing at infinity (or with suitable boundary conditions) we have
\[
\int d^n x \partial_\nu \left( \sqrt{-|g|} S^{\cdot \cdot \cdot} T^{\nu \cdot \cdot \cdot} \right) = 0,
\]
but the calculation above shows that this is the same as
\[
\int d^n x \sqrt{-|g|} D_\nu (S^{\cdot \cdot \cdot} T^{\nu \cdot \cdot \cdot}).
\]
from which the claim follows using the Leibnitz property of covariant derivatives.

4. Consider a spacetime with metric

$$ds^2 = dt^2 - dx^2$$

where

$$dx^2 = g_{ij}(t,x)dx^i dx^j$$

is an arbitrary Riemannian metric on what we may call space.

(a) Show that $x = \text{const}$ is a geodesic.

For the metric given we have

$$\Gamma^i_{00} = \frac{1}{2} g^{ij} (2 \partial_t g_{i0} - \partial_i g_{00}) = 0 .$$

Thus, setting $u^i = 0$ we find that the geodesic equation is $du^i/ds = 0$, establishing that our coordinates are comoving, i.e. $x = \text{const}$ is a geodesic.

(b) If $g_{ij}$ is $t$-independent, show that every geodesic follows a geodesic track through space.

If $g_{ij}$ is $t$-independent then so is the entire metric and time derivatives of the metric are identically zero. Then we have in addition to the above

$$\Gamma^i_{0j} = \frac{1}{2} g^{ik} (\partial_0 g_{kj} + \partial_j g_{0k} - \partial_k g_{0j}) = 0 .$$

The spatial components of the geodesic equation then reduce to

$$\frac{du^i}{ds} + \Gamma^i_{jk} u^j u^k = 0 ,$$

the equation for a geodesic of $g_{ij}$.

5. In this problem we will use our newfangled technology to solve an old familiar problem in the hope that this will help us understand the meaning of some of the things we are playing with. The problem is nonrelativistic projectile motion in a uniform gravitational field. The solution should be $d^2 z/dt^2 = -g$ with all other components of acceleration vanishing. We want to find the metric and Christoffel connection.
describing this background. To get these we start by constructing inertial (freely falling) coordinates. In this simplified problem we can in fact find \textit{globally} inertial coordinates, because what we are in fact describing is not real gravity but the fake gravity observed in an accelerating frame. The inertial coordinates are determined up to rotation and boost, but a simple choice is just \( z' = z + gt^2/2 \) with all other \( x'\mu = x^\mu \). In the inertial coordinates, of course, \( g'_{\mu\nu} = \eta_{\mu\nu} \) and Newton’s first law holds.

(a) Using the coordinate transformation properties of the metric, find \( g_{\mu\nu} \) and \( g^{\mu\nu} \) in the noninertial coordinates. To simplify your work you can note that the problem is essentially two-dimensional, anything to do with \( x, y \) is trivial.

To find the metric the easiest thing to do is set

\[
g_{\mu\nu}dx^\mu dx^\nu = ds^2 = g'_{\lambda\sigma}dx'^\lambda dx'^\sigma
\]

with

\[
dx'^\lambda = \frac{\partial x'^\lambda}{\partial x^\mu}dx^\mu.
\]

Expanding you can see that this is equivalent to our definition. In our case we have \( g' = \eta \) and the nontrivial coordinate transformation is \( z' = z + gt^2/2 \) so we find

\[
g_{\mu\nu}dx^\mu dx^\nu = ds^2 = dt^2 - (dz + gtdt)^2 = (1 - g^2t^2)dt^2 - 2gtdzdt - dz^2,
\]

where for simplicity I am suppressing the trivial \( x, y \) components. Thus

\[
g = \begin{pmatrix}
1 - g^2t^2 & 0 & 0 & -gt \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
-gt & 0 & 0 & -1
\end{pmatrix}.
\]

To find the inverse metric from the transformation laws we can use a similar construction with the Laplacian (Delambertian, really) operator replacing the interval

\[
g^{\mu\nu}\partial_\mu \partial_\nu = \partial^2 = g'^{\lambda\sigma} \partial'_\lambda \partial'_\sigma,
\]
with

\[ \partial'_\lambda = \left( \partial x^\mu / \partial x'^\lambda \right) \partial_\mu . \]

With \( z = z' - gt^2/2 \) as the nontrivial transformation we find here

\[ g^{\mu\nu} \partial_\mu \partial_\nu = (\partial_t - gt \partial_z)^2 - \partial_z^2 = \partial_t^2 - 2gt \partial_t \partial_z - (1 - g^2 t^2) \partial_z^2 . \]

Thus

\[ g^{-1} = \begin{pmatrix} 1 & 0 & 0 & -gt \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -gt & 0 & 0 & -1 + g^2 t^2 \end{pmatrix} . \]

(b) Check your results in (a) by verifying that \( g^{\mu\nu} g_{\nu\lambda} = \delta^\mu_\lambda \) as it must be. Restricting to the spatial components of the metric \( g_{ij} \) gives a description of a time-slice, or what we normally call space. What does the form of our metric tell you about the shape of space in this problem? Does that make sense?

Our metric does not break down in finite time as was suggested by an early version. The determinant \( |g| = -1 \) for all \( t \). Solving for the eigenvalues we find, in addition to the two obvious \(-1\) eigenvectors,

\[ \lambda_{1,2} = -\frac{g^2 t^2}{2} \pm \sqrt{\frac{g^4 t^4}{4} + 1} . \]

We see that as \( t \to \infty \) the positive (spacelike) eigenvalue approaches zero, meaning our \( z \) coordinate becomes null - a coordinate breakdown. We will see better examples of this soon.

The spatial metric \( g_{ij} = \delta_{ij} \) at all time shows that space has all of the symmetries of Euclidean three-dimensional space.

(c) Now, using the metric components you computed above, find the components \( \Gamma^\mu_{\nu\lambda} \) of the Christoffel connection in the noninertial frame.

Computing we find only one nontrivial connection component

\[ \Gamma^z_{tt} = g . \]
(d) Reobtain the answer to (c) by using the non-tensorial transformation properties of $\Gamma$ together with the fact that in the (global) inertial frame $\Gamma' = 0$ to find $\Gamma$ directly.

Starting with $\Gamma' = 0$ we find that only the non-tensorial part of the transformation equations is nonzero, leading to

$$\Gamma^\mu_{\nu\lambda} = -\frac{\partial x'^\sigma}{\partial x^\nu} \frac{\partial x'^\rho}{\partial x^\lambda} \frac{\partial^2 x^\mu}{\partial x'^\sigma \partial x'^\rho}.$$

The only nontrivial second derivative is $\frac{\partial^2 z}{\partial t'^2} = -g$ and with $t' = t$ we find

$$\Gamma^z_{\nu\lambda} = g \frac{\partial t'}{\partial x^\nu} \frac{\partial t'}{\partial x^\lambda} = g \delta_0^\nu \delta^0_\lambda.$$

(e) Now, finally, write down the equations for geodesics in the noninertial coordinates

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

and show that they lead to the expected result for nonrelativistic projectile motion.

The nontrivial geodesic equation is thus

$$\frac{d^2 z}{d\tau^2} + g \left( \frac{dt}{d\tau} \right)^2 = 0.$$

For nonrelativistic motion $t = \tau$ and we find the familiar result.

6. Consider the two-sphere with the metric $d\Omega = d\theta^2 + \sin^2 \theta d\phi^2$. You found the Christoffel connection for this in problem 2. Use this to perform the following exercise in parallel transport. We start with the tangent vector $u$ (of unit length) to the curve $\phi = 0$ (a line of longitude) at the North pole $\theta = 0$. This is then parallel transported, i.e. moved along the curves listed below so that

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\nu\lambda} \frac{dx^\nu}{ds} u^\lambda = 0,$$

where $s$ is any parameter (note that this equation is in fact invariant under reparameterizations of the curve) along the curve and we use the initial condition to solve this
finding a tangent vector \( u \) at any point along the curve. In our case, we transport \( u \) along these curves:

(a) Along the curve \( \phi = 0 \) from \( \theta = 0 \) to \( \theta = \theta_0 \).

(b) Along the curve \( \theta = 0 \) from \( \phi = 0 \) to \( \phi = \phi_0 \).

(c) Along the curve \( \phi = \phi_0 \) from \( \theta = \theta_0 \) back to the North pole \( \theta = 0 \).

Find the transported vector at the end point of each of these stages. Note that the final stage ends at the starting point, but we did not find the same vector we started with. Parallel transport along a closed curve can be nontrivial. By what angle has our initial vector been rotated when we bring it back to the pole? You may find it helpful to augment your calculations here by drawing the sphere and the transported tangent vectors.

We found the nontrivial components of the Christoffel connection above

\[
\Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta \quad \Gamma_{\theta\phi}^\phi = \cot \theta .
\]

The equations for parallel transport of a vector \( v \) are thus

\[
\frac{dv^\theta}{ds} - \sin \theta \cos \theta v^\phi \frac{d\phi}{ds} = 0
\]

\[
\frac{dv^\phi}{ds} + \cot \theta \left( v^\theta \frac{d\phi}{ds} + v^\phi \frac{d\theta}{ds} \right) = 0.
\]

We now apply these to our three segments. We begin with the components \( v^\theta = 1 \) and \( v^\phi = 0 \) at the pole. Note that while the coordinates we are using fail at the pole (metric has zero eigenvalue; direction of \( \theta \) is ill-defined, etc) I can write these components and they are meaningful in the context of the motion along the particular geodesic in (a).

(a) Along the line of longitude \( \phi = 0 \) we can use \( s = \theta \) (note that since we are not solving the geodesic equation the parameter \( s \) is undetermined, indeed the parallel transport equations hold for any parameter). Our equations become

\[
\frac{dv^\theta}{ds} = 0
\]

\[
\frac{dv^\phi}{ds} + \cot \theta v^\phi = 0.
\]
and the solution with the given initial conditions is simply $v^\theta = 1$ and $v^\phi = 0$. This is, of course, not surprising since the line of longitude is a geodesic and $v$ a tangent vector.

(b) Along the line of latitude $\theta = \theta_0$ we can use $s = \phi$. Our equations are now

\[
\frac{dv^\theta}{ds} - \sin \theta_0 \cos \theta_0 v^\phi = 0
\]
\[
\frac{dv^\phi}{ds} + \cot \theta_0 v^\theta = 0 .
\]

Plugging one into the other as usual for such systems we find

\[
\frac{d^2v^\theta}{ds^2} + \cos^2 \theta_0 v^\theta = 0
\]
\[
\frac{d^2v^\phi}{ds^2} + \cos^2 \theta_0 v^\phi = 0 .
\]

These are harmonic oscillator equations. The components of $v$ oscillate in $\phi$ with angular frequency $\cos \theta_0$. Note that as expected, at the equator ($\cos \theta_0 = 0$) the oscillations disappear (the equator is a geodesic) while near the poles ($\cos \theta_0 = \pm 1$) the angular frequency approaches one. Near enough to the pole we can find to a good approximation local flat coordinates including our entire line of latitude. In these coordinates parallel transport is simple ($\hat{\Gamma} = 0$) and the oscillation we are finding is simply the rotation of the $\hat{\theta}$ and $\hat{\phi}$ directions around a small circle about the origin in flat space.

With the results of (a) as our initial conditions we find our solution

\[
v^\theta(\phi) = \cos (\cos(\theta_0)\phi)
\]
\[
v^\phi(\phi) = -\csc(\theta_0) \sin (\cos(\theta_0)\phi) ,
\]

and evaluating these at $\phi_0$ yields the initial conditions for the next segment.

(c) Along the line of longitude $\phi = \phi_0$ we can use $s = \theta_0 - \theta$ and our equations become

\[
\frac{dv^\theta}{ds} = 0
\]
\[
\frac{dv^\phi}{ds} - \cot \theta v^\phi = 0 .
\]
These are linear ODEs and we can solve them; the second is solved by (recall $ds = -d\theta$)

$$\frac{dv^\phi}{v^\phi} = \cot \theta \, d\theta = -\frac{d\sin \theta}{\sin \theta}.$$ 

With our initial conditions we find

$$v^\theta = \cos (\cos(\theta_0)\phi_0)$$
$$v^\phi = -\sin (\cos(\theta_0)\phi_0) (\sin \theta)^{-1}.$$ 

We need not be alarmed by the divergence of $v^\phi$ as $\theta \to 0$ at the pole. This reflects the degeneration of our coordinates at this point. Indeed if we compute the length at any point along the curve we find

$$v^2 = (v^\theta)^2 + \sin^2 \theta (v^\phi)^2 = \cos^2 (\cos(\theta_0)\phi_0) + \sin^2 (\cos(\theta_0)\phi_0) = 1.$$

To find the angle by which our vector has rotated after the entire circuit we can use the fact that $v$ is normalized to unit length throughout to compute the inner product using the metric at the pole

$$\cos \alpha = v_i \cdot v_f = g_{kl} v^k_i v^l_f = g_{\theta \theta} v^\theta_f = \cos (\cos(\theta_0)\phi_0) .$$

This is obviously incorrect. Obviously because, for example, setting $\theta_0 = \pi/2$ so that there is no rotation along the equatorial segment we find $\alpha = 0$ whereas a moment’s thought should convince you that in this case $v_f$ the answer is clearly $\alpha = \phi_0$. Indeed, this little example shows the error in the method. At the pole, not only is the $\phi$ direction singular, the $\theta$ direction is ill-defined: any motion at the pole has a velocity proportional to $v_i$, so the expression of a vector in these coordinates is meaningless.

To avoid using a degenerate coordinate system, we should pick a better basis at the north pole. As mentioned in class, near the pole a convenient set of coordinates is given by

$$x = \sin \theta \cos \phi$$
$$y = \sin \theta \sin \phi .$$
In these coordinates the metric at the pole is simply \( ds^2 = dx^2 + dy^2 \). Tangent vectors transform as

\[
\begin{align*}
    v^x &= \cos \theta \cos \phi v^\theta - \sin \theta \sin \phi v^\phi \\
    v^y &= \cos \theta \sin \phi v^\theta + \sin \theta \cos \phi v^\phi .
\end{align*}
\]

In these coordinates, our initial vector at the pole was simply \( v_i = (1, 0) \) (approach the pole along \( \phi = 0 \) with \( v^\theta = 1 \)). Our final vector, after returning to the pole, is now (approaching along \( \phi = \phi_0 \))

\[
\begin{align*}
    v^x_f &= \cos \phi_0 \cos (\cos(\theta_0)\phi_0) + \sin \phi_0 \sin (\cos(\theta_0)\phi_0) = \cos ((1 - \cos(\theta_0))\phi_0) \\
    v^y_f &= \sin \phi_0 \cos (\cos(\theta_0)\phi_0) \cos \phi_0 \sin (\cos(\theta_0)\phi_0) = \sin ((1 - \cos(\theta_0))\phi_0) .
\end{align*}
\]

The squared length of this \( (v^x)^2 + (v^y)^2 \) agrees, of course, with our previous result for this. We now clearly have

\[
\alpha = (1 - \cos(\theta_0)) \phi_0 .
\]

We can make some sanity checks here. For \( \theta_0 = \pi/2 \) we find \( \alpha = \phi_0 \) as expected. For \( \theta_0 \to 0 \) we find \( \alpha = 0 \). A vanishingly small triangle produces essentially no rotation.

Interestingly, if you compute the area of the spherical triangle our closed path circles you will find it is precisely equal to \( \alpha \). Can you see why this is true?