1. Two particles freely orbit a star of mass $M$ at Schwarzschild radii $r_2 > r_1$ in the same plane and in the same sense. The particle at $r_2$ emits radiation with proper frequency $\nu_2$ which is then absorbed by the particle at $r_1$. Find the frequency at which the radiation is absorbed.

2. In class we defined the quantity

$$E = (1 - 2GM/r) \frac{dt}{d\lambda}$$

and showed it is conserved in free motion in Schwarzschild space. In this problem you will study this quantity to understand in what sense it corresponds to a measure of the orbiting object’s energy

(a) Show that to first approximation at large-$r$ we have

$$E \sim 1 + \frac{v^2}{2} - \frac{GM}{r},$$

where $v$ is the orbital velocity as observed by a static observer at the (large) radius $r$. Interpret this as the total energy per unit mass of the orbiting object.

(b) In general, show that if an observer with four-velocity $u^\mu$ observes a particle with rest mass $m$ moving at four-velocity $v^\mu$, the observer records a total energy of $mu \cdot v$. This is best understood by going to the instantaneous inertial frame of the observer. Use this to find the energy, as observed by a Schwarzschild static observer at radius $r$, of a particle moving by with four-velocity $u^\mu$. Note that this will not be given by $E(u)$, nor will it in general be conserved for a freely orbiting particle.

(c) Show that if this energy is completely converted to radiation and emitted radially outward, then absorbed at infinity, the total energy observed at infinity is $E(u)$. This is the sense in which the conserved quantity $E$ corresponds to “energy at infinity.”
3. Consider a spherically symmetric, static spacetime with a spherically symmetric, radially directed, time independent electric field vanishing at spatial infinity.

(a) Find the energy-momentum tensor corresponding to this field configuration (this will contain an undetermined function specifying the dependence of $E$ on $r$). The energy-momentum tensor for electromagnetic fields was given in HW 5 (problem 4) as

$$T^\mu_\nu = \frac{1}{4\pi} \left( - F^\mu_\sigma F^\nu_\sigma + \frac{1}{4} g^\mu_\nu F^\sigma_\rho F^\sigma_\rho \right).$$

(b) Use the spherical symmetry to follow the calculation of the Schwarzschild metric in class: solve Einstein’s equation

$$G^\mu_\nu \equiv R^\mu_\nu - \frac{1}{2} R g^\mu_\nu = 8\pi G T^\mu_\nu$$

for this example of an “electrovac” (i.e. vacuum except for electromagnetic fields) configuration. Your result will contain an undetermined constant. Use Gauss’s law to relate this to the charge $Q$ of the configuration, and show that the metric you found takes the Reissner-Nordstrom form

$$ds^2 = - \left( 1 - \frac{2GM}{r} + \frac{Q^2}{r^2} \right) dt^2 + \left( 1 - \frac{2GM}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 d\Omega.$$

(c) Find the period of a massive object in a circular orbit in this metric with circumference $2\pi R$ (note that this is simply the statement that $r = R$). What is the velocity of this orbiting object relative to a static observer at $r = R$?

(d) What are the electric and magnetic fields measured by the orbiting observer?

4. A small perturbation of an unstable circular orbit will grow exponentially in time. Show that a displacement $\delta r$ from the unstable maximum of the Schwarzschild effective potential for timelike geodesics will grow initially as

$$\delta r \propto e^{\tau/\tau_*},$$

where $\tau$ is the proper time along the particles trajectory and $\tau_*$ is a constant. Evaluate $\tau_*$ as a function of the initial orbital radius $R$. Explain its behavior as the radius of the orbit approaches $6GM$. 

5. Consider an observer in radial free fall towards a Schwarzschild black hole of mass $M$ with energy $E = 1$. If the observer is falling feet-first and is $h = 2m$ tall, find the magnitude of the tidal force stretching her, as a function of her instantaneous radial coordinate $r$. Perform your calculation to first order in $h$ and indicate where you expect this approximation to fail. Express your answer as a tidal acceleration in units of $g = 9.8 \text{ m/sec}^2$. 