1. Killing Vectors (named for Wilhelm Killing, nothing murderous about them) are associated to continuous symmetries of spacetime.

(a) Consider an infinitesimal change of coordinates

\[ x'(x) = x + \epsilon \xi(x) \]

where \( \xi^\mu \) is a vector field. Expand the tensor transformation properties of the metric

\[ g_{\mu'\nu'}(x') = \frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial x^\nu}{\partial x'^{\nu'}} g_{\mu\nu}(x(x')) \]

to first order in \( \epsilon \) and show that the metric is preserved iff

\[ D_{\{\mu} \xi_{\nu\}} = 0 \]

(b) Show that a Killing vector field \( \xi \) determines a conserved quantity. Along any geodesic, the quantity

\[ K = g_{\mu\nu} \xi^\mu u^\nu \]

where \( u^\mu \) is the four-velocity, is constant.

(c) Show that for a Killing vector \( \xi \), the Riemann tensor, Ricci tensor, and Ricci scalar satisfy

\[ D_\mu D_\sigma \xi^\rho = R^\rho_{\nu\mu\sigma} \xi^\nu \]
\[ D_\mu D_\sigma \xi^\mu = R_{\sigma\nu} \xi^\nu \]
\[ \xi^\lambda D_\lambda R = 0 \]

(d) We defined a stationary spacetime as one in which there is a set of coordinates in which the components of the metric are independent of time,

\[ ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu \]

Show that for any such metric, the vector \( \xi^\mu = \delta^{\mu\cdot0} \) is a timelike Killing vector.
(e) Use the results of previous parts, along with the contracted Bianchi identity
\[ D_{\mu} R^{\mu\nu} = \frac{1}{2} D^{\nu} R \]
to show that in any stationary spacetime the current
\[ J_{R}^{\mu} = \xi_{\nu} R^{\mu\nu} = 8\pi G \xi_{\nu} \left( T^{\mu\nu} - \frac{1}{2} T g^{\mu\nu} \right) \]
is conserved, i.e.
\[ D_{\mu} J_{T}^{\mu} = 0 . \]
The quantity
\[ E_{R} = \frac{1}{4\pi G} \int_{\Sigma} d^{3}x \sqrt{g^{(3)}} n_{\mu} J^{\mu} , \]
where \( \Sigma \) is a spacelike hypersurface, \( g^{(3)} \) the restriction of \( g \) to \( \Sigma \), and \( n_{\mu} \) is a unit normal vector to \( \Sigma \), is independent of the slice \( \Sigma \). We can take \( \Sigma \) to be a slice at constant \( t \) in the stationary coordinates, and what you will have shown is that \( E_{R}(t) \) is independent of \( t \). In a static spacetime, the orthogonal unit vector \( n \) can then be taken proportional to \( \xi \), so that \( n \cdot J \) is proportional to \( J^{0} \), yielding the form in which we are used to see conservation laws.

(f) Use the identities from part (b) to show that
\[ J_{R}^{\mu} = D_{\nu} (D^{\mu} \xi_{\nu}) , \]
and use this to show that \( E_{R} \) can be computed as an integral over the boundary \( \partial \Sigma \)
\[ E_{R} = \frac{1}{4\pi G} \int_{\partial \Sigma} d^{2}x \sqrt{g^{(2)}} n_{\mu} \sigma_{\nu} D^{\mu} \xi_{\nu} , \]
where \( \sigma \) is an outward pointing unit normal to the boundary. This boundary integral defines the Komar energy of the spacetime.

(g) To justify the name, let’s apply this reasoning to the Schwarzschild metric
\[ ds^{2} = \left( 1 - \frac{2GM}{r} \right) dt^{2} - \left( 1 - \frac{2GM}{r} \right)^{-1} dr^{2} - r^{2} d\Omega^{2} . \]
Use space at a fixed \( t \) as your slice \( \Sigma \), and use some care in defining the unit normals; the nontrivial contribution comes from the boundary at infinity.
2. The AdS Black Hole appeared in some discussions in class. In this problem we will try to learn some things about this interesting solution of Einstein’s equation with a negative cosmological constant

\[ G_{\mu\nu} = -\frac{3}{b^2} g_{\mu\nu}, \]

where the normalization of \( b \) is chosen so that the maximally symmetric AdS solution is obtained by restricting the flat metric on \( \mathbb{R}^{2,3} \) to the hyperboloid given by

\[ X_1^2 + X_2^2 - X_3^2 - X_4^2 - X_5^2 = b^2. \]

In class we used the more common notation \( \alpha^2 \) but we have other uses for that symbol here. In static coordinates we wrote this as

\[ ds^2 = (1 + \frac{r^2}{b^2}) dt^2 - (1 + \frac{r^2}{b^2})^{-1} dr^2 - d\Omega^2. \]

(a) Assume a static, spherically symmetric spacetime and find a one-parameter family of solutions to Einstein’s equation with a negative cosmological term, mimicking our derivation of the Schwarzschild solution. You can use this analogy to label the parameter (constant of integration) although this is less straightforward than in asymptotically flat Schwarzschild spacetime. Your solution will have a singularity at \( r = 0 \) as well as an event horizon.

(b) This solution shares many of the symmetries of the Schwarzschild spacetime. As a static spacetime, it exhibits the timelike Killing vector \( \xi \) from the previous problem. Spherical symmety will also be reflected in Killing vectors. Show that the vector \( \mathcal{R}^\mu = \delta^{\mu\phi} \) is a Killing vector. Find the conserved quantities \( E, L \) associated to the two Killing vectors.

(c) Since the spacetime is not asymptotically Minkowski, \( E \) will not be “energy per unit mass at infinity,” nor are static observers inertial at infinity. \( E \) will be the effective energy as measured by a static observer at a radial coordinate \( r = r_E \). Find \( r_E \).

(d) Find the radial coordinate \( r_* \) at which a circular null orbit exists. The orbit will be unstable, so small disturbances will grow exponentially with time. Find the characteristic time for this growth and the period of the orbit (as measured by a static observer at \( r_* \)) Is it properly an orbit?
3. As we discussed in class, our present universe is characterized by $\Omega_\Lambda = 0.7; \Omega_M = 0.3; \text{ and } \Omega_R = 8.5 \times 10^{-5}$. We also measure a Hubble constant of $H_0^{-1} = 13.4 \times 10^9 \text{ yr}$.

(a) Cosmologists conclude from this that the universe is “flat,” despite the obvious fact that the Riemann tensor of the FRW metric describing our universe is nonzero. Explain what is meant by “flat” in this context.

(b) Assume that the expansion of the universe in the present epoch is dominated by the dark energy, and use the appropriate solution of the Friedmann equation to find the time $t_M$ at which the density of matter was equal to the density of dark energy. Find the redshift factor corresponding to this time.

(c) Assume further that prior to $t_M$ the expansion was matter dominated. Use the appropriate solution to find the time $t_R$ at which the energy density in radiation was equal to the energy density in (cold) matter. What was the temperature of the radiation at this time? Find the redshift factor corresponding to this time.

(d) The earliest known galaxies are characterized by a redshift factor $z = 6$. If the previous answers divide the history of the cosmos into eras of radiation, matter, and dark energy (in chronological order), in what era were galaxies formed? At what time?