1. In the spectrum of the galaxy NGC 4839, the K line of singly ionized calcium is observed to have a wavelength of $\lambda = 403.2$ nm. In the lab, this spectral line is observed at $\lambda_0 = 393.3$ nm.

(a) Find the redshift factor $z = (\lambda - \lambda_0)/\lambda_0$ for NGC 4839.

$$z = \frac{\lambda - \lambda_0}{\lambda_0} = \frac{9.9}{393.3} = 0.025 .$$

(b) Using the formula $1 + z = \sqrt{\frac{c+v}{c-v}}$ find the velocity with which NGC 4839 is receding from us.

From $1 + z = \sqrt{\frac{c+v}{c-v}}$ we find

$$(1 + z)^2 = \frac{c + v}{c - v} = \frac{1 + v/c}{1 - v/c} ,$$

or rearranging

$$\frac{v}{c} = \frac{(1 + z)^2 - 1}{(1 + z)^2 + 1} = 0.0247 .$$

I have kept an extra digit here to show how this differs – slightly – from the non-relativistic expression $z = v/c$. This corresponds to a speed of

$$v = 0.025c = 7.5 \times 10^6 \frac{\text{m}}{\text{s}} = 7500 \frac{\text{km}}{\text{s}} .$$

(c) Using the value $H_0 = 73 \text{ km/s/Mpc}$ find the distance to NGC 4839.

Setting $v = H_0d$ we find

$$d = \frac{v}{H_0} = \frac{7500}{73} \text{ Mpc} = 103\text{ Mpc} .$$

(d) Interpreting $z$ as a *cosmological* redshift, find the factor by which distances have grown since the light was emitted.
Seen as a cosmological redshift $1 + z$ is the factor by which distances have grown since the light was emitted. Wavelength of light scales with other distances. Since this light was emitted all distances have grown by a factor of 1.025 (by 2.5%).

(e) Assuming a uniform expansion with scale factor $a(t) = 1 + H_0 t$ where $t = 0$ is defined as the present moment, find a relation between the scale factor $a(t)$ at the time the light was emitted and the redshift $z$. Use this to find the time $t$ at which the light was emitted (you should find a negative answer, since this is clearly in the past!).

We have clearly set the scale factor in the present ($t = 0$) to one here, so scale factor at time of emission $t$ satisfies $a(t)(1 + z) = 1$. After stretching by $1 + z$ we arrive at present distances. This gives

$$a(t) = \frac{1}{1 + z}.$$ 

Equating this to the uniform expansion expression we have

$$1 + H_0 t = \frac{1}{1 + z},$$

or

$$H_0 t = \frac{1}{1 + z} - 1 = -\frac{z}{1 + z}.$$ 

We thus have

$$t = -\frac{z}{1 + z} H_0^{-1} = -0.024 \times 13.3 \times 10^9 = -3.24 \times 10^8 \text{ yr}.$$ 

We are observing light from 324 million years ago. Note I used the value for the inverse Hubble constant

$$H_0^{-1} = (73 \text{km/s/Mpc})^{-1} = (73/3.0857 \times 10^{19})^{-1} \text{s} = 4.2 \times 10^{17} \text{s} = 1.33 \times 10^{10} \text{ yr}.$$ 

(f) Use this time and the speed of light to find another estimate of the distance to NGC 4839. Compare to your answer in (c).

In the 324 years since the light was emitted, it will have traveled

$$d = 3.24 \times 10^8 \times 3.17 \times 10^7 \times 3 \times 10^8 = 3.06 \times 10^{24} \text{ m} = 100 \text{ Mpc}.$$
Rounding several calculations to two significant digits, an error of 3% is reasonable. The two results are consistent.

2.

(a) Repeat steps (b)-(f) above for the galaxy RD 1 observed at a redshift $z = 5.34$.

Here we find

$$\frac{v}{c} = \frac{(1 + z)^2 - 1}{(1 + z)^2 + 1} = 0.95,$$

or

$$v = 0.95c = 2.85 \times 10^5 \text{km/s},$$

hence

$$d = \frac{v}{H_0} = \frac{285000}{73} = 4 \text{Gpc}.$$

This light has been stretched, and with it all distances, by a factor of $1 + z = 6.34$, which means it was emitted at time

$$t = -\frac{z}{1 + z} H_0^{-1} = -1.1 \times 10^{10} \text{yr}.$$

We see RD1 as it was 11 billion years ago. In this time light would have traveled

$$d = ct = 1.1 \times 10^{10} \times 3.17 \times 10^7 \times 3 \times 10^8 = 1.1 \times 10^{26} \text{m} = 3.4 \text{Gpc}.$$

The errors here are more significant. At these times our assumption of uniform expansion is no longer accurate, nor is the linear form of Hubble’s law. We will see what replaces these in class.

(b) At the time the light we observe today was emitted by RD 1 what was the average density of matter in the universe?

If distances have scaled by a factor of 6.34, volumes have scaled by $6.34^3 = 255$. Since the total energy in matter has been conserved over this time, the density of matter was then 255 times larger than its present value, or

$$\rho_M = 255 \rho_{M,0} = 255 \times 2.4 \times 10^{-27} = 6.12 \times 10^{-25} \text{kg/m}^3.$$
(c) At the time the light we observe today was emitted by RD 1 what was the average density of energy in radiation in the universe?

When distances scale by 6.34 volumes scale by 255. The number of photons has been conserved over this period, since they have basically been propagating freely in a vacuum. But the wavelength of each photon has also been scaled, so that the energy per photon has decreased by a factor of 6.34 due to the cosmological redshift. The energy density in radiation 11 billion years ago was thus larger than today’s value by $6.34^4 = 1616$ or

$$\rho_R = 1600 \times 4.6 \times 10^{-31} = 7.4 \times 10^{-28} \text{ kg/m}^3.$$ 

(d) Assuming that the density of dark energy in the universe is constant (cosmological constant) how did the density of dark energy compare to the densities of matter and radiation at the time the light from RD 1 was emitted?

Our best estimate for the density of dark matter is

$$\rho_\Lambda = 3.2\rho_{M,0} = 7.6 \times 10^{-27} \text{ kg/m}^3.$$ 

Today this is three times the matter density and many times larger than the radiation density. 11 billion years ago, matter density was 100 times more than this, and radiation density only 10 times less. When the light we see from RD 1 was emitted, the universe was still matter-dominated.