1. Sound waves travel through air at approximately $330 \frac{m}{sec}$. Lightning is an electrical discharge (spark) that emits a pulse of light and a loud sound as air is heated and expands explosively. People often count off the time from the instant they see a lightning strike until they hear the thunder, and use the length of the time interval to estimate their distance from the strike’s location. Can you understand how this works? How many seconds would you count off if the strike is 10 km. away?

The light flash (lightning) travels through air at essentially $c$ and so reaches our eyes in a time indistinguishable from instantaneous detection. (Remember, light can circumnavigate the equator in 0.13 sec and you are not likely to see lightning that far.) Thunder (sound) produced simultaneously travels at the speed of sound and reaches our ear after a delay $T$ given (neglecting the transit time for light) by

$$T = \frac{D}{v_S}$$

where $v_S = 330 \frac{m}{sec}$. For $D = 10^4 m$ we find

$$T = 3 \text{ sec}.$$ 

2. The star Aldebaran, in Taurus, has a surface temperature of $3850^\circ$K. What is its wavelength of maximum emission? What color would you expect it to be? Compare the flux of radiation at Aldebaran’s surface to that on the surface of the Sun. Aldebaran has a radius 219 times that of the Sun. How does it compare to the Sun in total luminosity (power radiated)? (The surface area of a sphere is $4\pi R^2$ where $R$ is the radius.)

The wavelength of maximum emission is given by Wien’s Law

$$\lambda_m = \frac{0.0029}{T} = 7.5 \times 10^{-7} \text{ m} = 750 \text{ nm}$$
since a nanometer is $10^{-9}$ m. This is red light, and Aldebaran (the bull’s eye) is indeed red, as we saw at our observation sessions. The flux at the star’s surface is given by the Stefan-Boltzmann law

$$F = \sigma T^4.$$  

To compare this to the flux at the surface of the Sun we do not need to plug in the value of $\sigma$, it is enough to note the ratio

$$\frac{F_{\text{Aldebaran}}}{F_{\odot}} = \left(\frac{T_{\text{Aldebaran}}}{T_{\odot}}\right)^4 = \left(\frac{3850}{5800}\right)^4 = 0.19.$$  

Each square meter of Aldebaran’s surface radiates about five times less energy than a square meter of the Solar surface. On the other hand, Aldebaran is larger than the Sun. Since its radius is 219 times larger, its surface area is larger by

$$\frac{A_{\text{Aldebaran}}}{A_{\odot}} = \left(\frac{R_{\text{Aldebaran}}}{R_{\odot}}\right)^2 = 219^2 = 4.8 \times 10^4$$  

so that the total luminosity

$$L = FA$$  

is related to that of the Sun by

$$\frac{L_{\text{Aldebaran}}}{L_{\odot}} = \left(\frac{F_{\text{Aldebaran}}}{F_{\odot}}\right) \times \left(\frac{A_{\text{Aldebaran}}}{A_{\odot}}\right) = 9100.$$  

The red giant Aldebaran is a far more luminous object than the Sun.

3. Black holes are objects whose gravity is so strong nothing can escape, not even light. We can’t “see” a black hole. But we can see stuff falling into one. As we shall see later, material falling into a black hole is compressed and heated. Calculations suggest it is heated to a temperature of $10^6$ degrees K. At what wavelength would this stuff radiate? Could we detect the radiation on Earth?

It is important to note that the radiation in question is produced outside the black hole and we can, and do, detect it. Approximating the infalling matter as a blackbody, it will radiate maximally at the wavelength given by Wien’s law

$$\lambda_m = \frac{0.0029}{T} = 2.9 \text{ nm}.$$  

This is an X-Ray wavelength, and this radiation can not be detected on Earth because X-Rays are absorbed in air within a few centimeters. Indeed, the first detection of a black hole occurred when a photographic plate sensitive to X-Rays was sent into space on a rocket in 1962.

4. In observations of the star **Megrez** in the Big Dipper (part of *Ursa Major*) the $H_\beta$ line of Hydrogen is observed at a wavelength of 486.112 nm. In the laboratory, the wavelength of this line is measured to be 486.133 nm. Is the star moving towards Earth or away from us? At what speed?

Since we detect the radiation with a wavelength $\lambda$ shorter than the wavelength at which it appears in the star’s rest frame (where the cosmological principle tells us it is identical to what we measure in the lab), the star is moving **towards** us. The speed at which it is nearing is given by

$$v/c = \frac{\lambda - \lambda_0}{\lambda_0} = -4.3 \times 10^{-5} .$$

Megrez is nearing us at a speed of

$$v = 4.3 \times 10^{-5} \times 3 \times 10^8 = 1.3 \times 10^4 \text{ m sec}^{-1} .$$

5. Let’s try to find the **Roche limit**, the distance from a planet at which a moon would be broken apart by tidal forces. We will make a few simplifying assumptions along the way, and at one point I will help you with the math, but we can get an answer and hopefully learn something.

We will be considering a planet of radius $R$ and a moon of radius $r$ orbiting the planet at a distance (center to center) of $D$. If $D$ is sufficiently small, the moon will be ripped apart by the tidal forces of the planet. Our goal is to find the value of $D$ at which this happens.

(a) If the planet has mass $M$, write an expression for the gravitational acceleration of an object at a distance $D$ from the planet’s center. This is the acceleration with which the moon will “fall” as it orbits the planet.

The moon, or any other object located on the Moon’s orbit on which the only force is the gravitational attraction of the planet, will fall toward the planet with an acceleration determined from Newton’s law

$$a(D) = \frac{GM}{D^2} .$$
(b) Write an expression for the gravitational acceleration at a distance from the planet of $D - r$. This corresponds to the position of a point on the moon’s surface facing the planet.

At a distance $D - r$ from the planet, an object on which no force acts other than the planet’s gravitational attraction will fall toward the planet with an acceleration

$$a(D - r) = \frac{GM}{(D - r)^2}.$$ 

(c) The tidal acceleration exerted by the planet is the difference of these two expressions. If $r$ is much less than $D$, we can simplify this difference by using an approximation. This is the statement that

$$\frac{1}{(1 - x)^2} \sim 1 + 2x,$$

to a precision that improves as $x$ gets smaller. Check this approximation by evaluating the two sides for $x = 0.3, 0.1, 0.03, 0.01$ and finding the difference.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\frac{1}{(1-x)^2}$</th>
<th>$1 + 2x$</th>
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</tr>
<tr>
<td>0.1</td>
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<tr>
<td>0.01</td>
<td>1.0203</td>
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</tr>
</tbody>
</table>

(d) Use the simplifying equation of (c) to write an expression for the answer to (b), using $r/D$ as the small number $x$. Use this simplified version to write the tidal acceleration on the moon’s surface.

We can now approximate

$$a(D - r) = \frac{GM}{(D - r)^2} = \frac{GM}{D^2(1 - r/D)^2} = \frac{GM}{D^2} \left(\frac{1}{(1 - r/D)^2}\right) \sim \frac{GM}{D^2} \left(1 + 2r/D\right).$$

The tidal acceleration is the difference between this value (the force per unit mass on an object on the moon’s surface) and the acceleration $a(D)$ with which the moon is in fact falling as it orbits the planet. It expresses the amount by which the apparent weight of an object of mass 1 kg located on the moon’s surface on
the side facing the planet is reduced due to the planet’s gravitational force. Note
that this is not the force applied by the planet, but only the residual difference
after taking the moon’s free fall into account. Our approximation now simplifies
things

\[ a_{\text{tidal}} = a(D - r) - a(D) \sim \frac{GM}{D^2} (1 + 2r/D - 1) = \frac{2GMr}{D^3}. \]

(e) Now, the Roche limit will be the distance \( D \) at which this is equal to the gra-
\vitational acceleration at the moon’s surface due to the moon’s mass. For a moon
of mass \( m \) and radius \( r \), write an expression for this acceleration.

Neglecting tidal effects the gravitational acceleration on the surface of a moon of
mass \( m \) and radius \( r \) is

\[ a_g = \frac{Gm}{r^2}. \]

The tidal acceleration expresses the reduction in this. At the Roche limit the net
weight of an object vanishes. A gravitationally bound moon will thus disintegrate
(ultimately forming a ring) if it comes within this distance. This occurs when the
reduction due to tidal effects is equal to an object’s weight, i.e.

\[ a_{\text{tidal}} = a_g. \]

Inserting our expressions above

\[ \frac{2GMr}{D^3} = \frac{Gm}{r^2} \]

we can solve for

\[ D = \left( \frac{2M}{m} \right)^{1/3} r. \]

(f) The expressions look a bit more elegant if we rewrite them after expressing the
masses \( M \) and \( m \) in terms of the radii \( R \) and \( r \). The volume of a sphere of radius
\( R \) is

\[ V = \frac{4\pi}{3} R^3. \]

If the planet has an average density \( \rho_P \) and the moon an average density \( \rho_m \),
write the answers to (d) and (e) with the masses expressed in terms of the radii.
Finally, set the two accelerations equal and solve for the value of $D$ at which moons start to fall apart. Note that to this approximation $r$ drops out of your expressions. You should find an expression for $D$ in terms of $\rho_P$, $\rho_m$, and $R$. Once a moon starts to fall apart at the surface it will keep on breaking up until there is nothing left but a ring!

We now write the masses in terms of density and volume

$$M = \frac{4\pi R^3}{3} \rho_p \quad m = \frac{4\pi r^3}{3} \rho_m$$

whence

$$\frac{M}{m} = \left(\frac{R}{r}\right)^3 \left(\frac{\rho_p}{\rho_m}\right).$$

This yields in our expression for the Roche limit

$$D = \left(\frac{2\rho_p}{\rho_m}\right)^{1/3} R.$$

(h) The cool thing about an algebraic expression like the one we have now is that we can plug the numbers into it for various situations of interest to get quick answers. How close would Earth’s moon be able to come before it falls apart? The Moon’s average density is approximately 0.6 that of Earth.

Plugging in $\rho_m/\rho_p = 0.6$ we find

$$D = \left(\frac{2}{0.6}\right)^{1/3} R_{\oplus} = 1.5 R_{\oplus} = 9530 \text{ km}.$$

This is 40 times closer than the Moon’s present distance. When it first formed (if it did) at 0.1 its present distance it was indeed just outside the Roche limit.

(i) Saturn’s average density is only 0.126 of Earth’s density. Saturn’s A ring has an outer radius of 137,000 km. Comparing this to the radius of the planet, 60,268 km, what would the minimal density be of a shepherd moon orbiting inside the A ring (in terms of Earth’s density)? Would Earth’s moon survive in the A ring?
Since we wrote expressions, we have at the critical point, where a moon starts to fall apart

\[ D = \left( \frac{2 \rho_p}{\rho_m} \right)^{1/3} R \]

or

\[ \rho_m = 2 \left( \frac{R}{D} \right)^3 \rho_p = 2 \left( \frac{60286}{137000} \right)^3 \times 0.126 = 0.021 . \]

A moon with density as small as 0.021 that of Earth will survive in the A ring. Earth’s moon, with its density of 0.6 that of Earth, would surely survive. This explains the ability of shepherd moons to survive within the rings. These rocky objects are dense enough to withstand the tidal forces. Lighter ices cannot coalesce in this region so form rings.

6. In class, I talked about the fact that chemical forces, while much stronger that gravitational forces, act on the boundary or on the surface of materials rather than on the bulk. This means that for small objects, chemical forces vastly dominate gravitational forces, but for sufficiently large objects, gravity rules. Let’s see how this works in an example. Imagine that some tidal forces threaten to rip apart our beloved Earth. In an attempt to save a favorite mountain, we tunnel underneath the mountain and glue it down with a powerful superglue. In this problem, we’ll ask how large a mountain could we hold down.

To simplify matters, let’s take our mountain to be a regular tetrahedron (a triangular pyramid) with all six sides of equal length \( a \). Your geometry text of long ago may have taught you this, but I had to recalculate so I will tell you that the volume of the mountain and the area of each of its four triangular sides are given by

\[ A = \frac{\sqrt{3}}{4} a^2 \]

\[ V = \frac{\sqrt{2}}{12} a^3 . \]

We will assume our mountain is made of rock with a density of \( \rho = 3000 \text{ kg/m}^3 \).

(a) Compute the mass of the mountain and hence the gravitational force with which Earth attracts it (you can use the acceleration of gravity \( g = 9.8 \text{ m/sec}^2 \) here), in terms of \( a \).
The mass is given by

\[ M = \rho V \]
\[ = \rho \left( \frac{\sqrt{2}}{12} a^3 \right) \]
\[ = \frac{\sqrt{2}}{12} \rho a^3. \]

The gravitational force is simply

\[ F_g = M g \]
\[ = \left( \frac{\sqrt{2}}{12} \rho a^3 \right) g \]
\[ = \frac{\sqrt{2}}{12} (3000 \text{ kg/m}^3) (9.8 \text{ m/s}^2) a^3 \]
\[ = 3465 \times a^3 \text{ N}. \]

This is the tidal force it would take to lift the mountain off Earth against the Earth’s gravitational pull.

(b) The strongest superglue for which I could find a spec has a bonding strength of 2000 psi. This means that to rip up a square inch (about 6.45 × 10^{-4} \text{ m}^2) of the glue requires a force of 2000 lb. A pound is a unit of weight (hence force, not mass) equal to about 4.42 N. If we use this adhesive to glue one face of the pyramid to the bedrock, compute (in terms of \( a \) once again) the force it would take to rip the mountain up.

First we convert the units:

\[ F = 2000 \text{ lb} = 2000 \times 4.42 \text{ N} = 8.84 \times 10^3 \text{ N} \]

while

\[ 1 \text{ in}^2 = 6.45 \times 10^{-4} \text{ m}^2. \]

From the specification, we need a force \( F \) to rip up an area of 1\text{ in}^2 or 6.45 ×
10^{-4} \text{ m}^2. \text{ To rip up our mountain we need to rip an area } A, \text{ requiring a force of }

\[ F_A = \frac{A}{6.45 \times 10^{-4}} \times F \]

\[ = \frac{A}{6.45 \times 10^{-4}} \times 8.84 \times 10^3 \text{ N} \]

\[ = 1.371 \times 10^7 A \]

\[ = 1.371 \times 10^7 \left( \frac{\sqrt{3}}{4} \right) a^2 \]

\[ = 5.937a^2 \times 10^6 \text{ N}. \]

(c) Find the value of \(a\) above which the mountain’s weight (your answer in (a)) exceeds the bonding strength of the glue (your answer to (b)). This is the typical size for rocks above which chemical bonds are no longer comparable to gravitational forces. Mountains larger than this are held in place by gravity, rather than by the mechanical strength of rock.

We find \(a\) by setting \(F_g\) equal to \(F_A\),

\[ F_g = F_A \]

\[ 3465 \times a^3 \text{ N} = 5.937a^2 \times 10^6 \text{ N} \]

\[ a = 1713 \text{ m} \]

For objects smaller than this, the force holding them to Earth is chemical. To pick them up you mostly need to rip up rock. Objects larger than this size are held down mostly by their weight. Since tidal forces are proportional to mass, hence volume, we conclude that tidal forces just strong enough to rip apart the Earth would break rock into chunks about 1 km across (good-sized mountains) at which point chemical bonds would prevent further destruction. Presumably a similar calculation with ices would correctly predict the size of ring matter in Saturn’s rings. This corrects the statement above: a moon inside the Roche limit breaks up into chunks held together by chemical bonds; as we saw there, a gravitationally bound object of any size will break up.