1. (20 pts) In each of the following state whether the result is a vector, or a scalar, or if the expression makes no sense. As usual a bold letter denotes a vector and a non-bold letter represents a scalar.

1. \( \mathbf{a} + \mathbf{b} \times \mathbf{c} \) Vector
2. \( \mathbf{a} + \mathbf{b} \cdot \mathbf{c} \) Vector+Scalar = Nonsense
3. \( (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - |\mathbf{b}| \mathbf{a} \) Vector
4. \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \) Vector
5. \( \frac{d}{dt} \mathbf{r} + \mathbf{r} \cdot \mathbf{a} \) Vector+Scalar = Nonsense
6. \( \frac{d}{dt}(\mathbf{r} \cdot \mathbf{a}) + |\mathbf{a}|^2 \) Scalar
2. (15 pts) Find the equation for the plane passing through (3, 3, -1) which is perpendicular to the two planes $x + y = 2$ and $2x + z = 10$.

The plane $x + y = 2$ can be written $\langle 1, 1, 0 \rangle \cdot \mathbf{r} = 2$ and so has normal vector $\mathbf{n}_1 = \langle 1, 1, 0 \rangle$. Similarly $2x + z = 10$ has normal vector $\mathbf{n}_2 = \langle 2, 0, 1 \rangle$.

The vector $\mathbf{n}_3 = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, -1, -2 \rangle$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. Thus the desired plane is $x - y - 2z = d$ for some value of $d$. Since it passes through $(3, 3, -1)$ we know that the equation must be

$$x - y - 2z = 2.$$
3. (15 pts) Find the volume of the tetrahedron with vertices $P(1, 3, -2)$, $Q(2, 8, 8)$, $R(1, 1, 1)$ and $S(1, 0, 0)$.

The volume of the parallelepiped generated by $SP$, $SQ$ and $SR$ is given by the magnitude of the triple scalar product of these vectors. This product is given by

$$\begin{vmatrix} 0 & 3 & -2 \\ 1 & 8 & 8 \\ 0 & 1 & 1 \end{vmatrix} = -5.$$ 

The volume of the tetrahedron is $\frac{1}{6}$ of this, i.e., $\frac{5}{6}$. 
4. (15 pts) Find the minimum distance between the line 

\[ y = z = 0 \]

and the line of intersection between the two planes \( x - y + 2 = 0 \) and \( x + 2y - z + 3 = 0 \).

Let \( L_1 \) be the first line which, in vector form, can be written

\[ \mathbf{r} = \mathbf{r}_1 + \mathbf{v}_1 t = \langle 0, 0, 0 \rangle + (1, 0, 0)t. \]

The second line lies in \( x - y + 2 \) and \( x + 2y - z + 3 = 0 \) and so must be in the direction given by \( \langle 1, -1, 0 \rangle \times \langle 1, 2, -1 \rangle = \langle 1, 1, 3 \rangle \). To find a point on this line we may set, say, \( y = 0 \) and solve \( x - y + 2 \) and \( x + 2y - z + 3 = 0 \) as simultaneous equations. This yields the point \((-2, 0, 1)\). In other words, \( L_2 \), is given by

\[ \mathbf{r} = \mathbf{r}_2 + \mathbf{v}_2 t = \langle -2, 0, 1 \rangle + (1, 1, 3)t. \]

The shortest distance can then be computed from

\[ d = \frac{\mathbf{v}_1 \times \mathbf{v}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{v}_1 \times \mathbf{v}_2|} \]

\[ = \frac{1}{\sqrt{10}} \]

as we discussed in class.
5. (15 pts) If we take a vector of the form $\langle a, b, c \rangle$ and rotate it $90^\circ$ around the $z$-axis it becomes the vector $\langle -b, a, c \rangle$. If the two three-dimensional vectors $\mathbf{u}$ and $\mathbf{v}$ are both rotated $90^\circ$ around the $z$-axis, what is the resulting transformation on $\mathbf{u} \times \mathbf{v}$?

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$\mathbf{u} \times \mathbf{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle.$$

Let $\mathbf{u}' = \langle -u_2, u_1, u_3 \rangle$ and $\mathbf{v}' = \langle -v_2, v_1, v_3 \rangle$. Then

$$\mathbf{u}' \times \mathbf{v}' = \langle -(u_3v_1 - u_1v_3), u_2v_3 - u_3v_2, u_1v_2 - u_2v_1 \rangle.$$

That is, $\mathbf{u}' \times \mathbf{v}'$ is $\mathbf{u} \times \mathbf{v}$ rotated by $90^\circ$ around the $z$-axis. This had better be true of course given the geometrical interpretation of the cross product!
6. (20 pts) Suppose a parametrized curve is given by \( \mathbf{r} = \langle t - \sin t, 1 - \cos t \rangle \). Find the unit tangent vector \( \mathbf{T} \), the unit normal vector \( \mathbf{N} \), and the curvature \( \kappa \) at any point as a function of \( t \). (You may want to recall the useful formulas \( \sin 2\theta = 2 \sin \theta \cos \theta \) and \( \cos 2\theta = 1 - 2 \sin^2 \theta \) in order to simplify your working.)

\[
\mathbf{r} = \langle t - \sin t, 1 - \cos t \rangle,
\]
and so

\[
\frac{d\mathbf{r}}{dt} = \langle 1 - \cos t, \sin t \rangle.
\]

Thus

\[
\frac{ds}{dt} = \left\| \frac{d\mathbf{r}}{dt} \right\| = \sqrt{2 - 2 \cos t} = 2 |\sin \frac{t}{2}|.
\]

To avoid confusion over signs let us assume \( 0 < t < 2\pi \) and so \( \sin \frac{t}{2} > 0 \). (The signs will need to be adjusted for the case \( 2\pi < t < 4\pi \).)

We then have

\[
\mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\langle 1 - \cos t, \sin t \rangle}{2 \sin \frac{t}{2}} = \frac{\langle 2 \sin^2 \frac{t}{2}, 2 \cos \frac{t}{2} \sin \frac{t}{2} \rangle}{2 \sin \frac{t}{2}} = \langle \sin \frac{t}{2}, \cos \frac{t}{2} \rangle.
\]

Now,

\[
\frac{d\mathbf{T}}{ds} = \frac{\langle \cos \frac{t}{2}, -\sin \frac{t}{2} \rangle}{2 \sin \frac{t}{2}} = \frac{1}{2} \langle \cot \frac{t}{2}, -1 \rangle.
\]

This gives

\[
\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{1}{2 \sin \frac{t}{2}}.
\]
and

\[ N = \frac{1}{\kappa} \frac{dT}{ds} = \langle \cos \frac{t}{2}, -\sin \frac{t}{2} \rangle. \]

(Actually since we are in only two dimensions, the normal vector could have been determined purely from the fact it is perpendicular to \( T \).)