1. (17 pts) A sphere is made of a material which has density given by

\[ \delta = 1 + \rho^2, \]

where \( \rho \) is the distance from the center. What is the mass of such a sphere of radius \( R \)?

Use spherical coordinates \((\rho, \theta, \phi)\).

\[
m = \iiint \delta \, dV
= \int_0^\pi \int_0^{2\pi} \int_0^R (1 + \rho^2) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi
= 2\pi \int_0^\pi \sin \phi \, d\phi \int_0^R (\rho^2 + \rho^4) \, d\rho
= 4\pi \left( \frac{R^3}{3} + \frac{R^5}{5} \right).
\]
2. (18 pts)

a) Show that \( \mathbf{F} = (e^y, xe^y, 1) \) is conservative by finding a scalar field \( \phi \) such that \( \mathbf{F} = \nabla \phi \).

b) Hence evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \),

where \( \mathbf{F} \) is as above and \( C \) is the curve \( \mathbf{r} = (t, \log(2 - t^2), 1/(t + 1)) \) with \( 0 \leq t \leq 1 \).

a) Since \( \partial \phi / \partial x = e^y \), we see that

\[
\phi = xe^y + f_1(y, z)
\]

for some function \( f_1 \). Similarly we see that

\[
\phi = xe^y y + f_2(x, z) \\
\phi = z + f_3(x, y)
\]

Thus we may put

\[
\phi = xe^y + z.
\]

(You can add any constant to this but the question asked for any scalar field \( \phi \) so we may set this constant to zero.)

b)

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla \phi \cdot d\mathbf{r} = \int_C d\phi = \phi(\text{end}) - \phi(\text{start}).
\]

At the start point \( \mathbf{r} = (0, \log 2, 1) \) and so \( \phi(\text{start}) = 1 \). At the end point \( \mathbf{r} = (1, 0, \frac{1}{2}) \) and so \( \phi(\text{end}) = \frac{3}{2} \).

Thus

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}.
\]
3. (25 pts) A sphere of radius $R$ centered at the origin has a hole of diameter $R$ drilled out of it in the shape of a cylinder. This cylinder has equation $r = R \cos \theta$ in cylindrical coordinates. What is the volume of the part of the sphere which remains.

Use cylindrical coordinates $(r, \theta, z)$. The $z = 0$ plane slice looks like:

The volume is given by the triple integral, where a volume element is given by $dV = r \, dz \, dr \, d\theta$. In cylindrical coordinates the equation of the sphere is $r^2 + z^2 = R^2$.

Let’s compute the volume of the material drilled out. This is given by

$$
4 \int_0^\frac{\pi}{2} \int_0^{R \cos \theta} \int_0^{\sqrt{R^2 - r^2}} r \, dz \, dr \, d\theta = 4 \int_0^\frac{\pi}{2} \int_0^{R \cos \theta} \sqrt{R^2 - r^2} \, dr \, d\theta
$$

$$
= 2 \int_0^{\frac{\pi}{2}} \int_{R^2 \sin^2 \theta}^{R^2} \sqrt{u} \, du \, d\theta
$$

$$
= \frac{4}{3} \int_0^{\frac{\pi}{2}} R^3 (1 - \sin^3 \theta) \, d\theta
$$

$$
= \frac{2}{3} \pi R^3 - \frac{4}{3} \int_0^1 (1 - v^2) \, dv
$$

$$
= \frac{2}{3} \pi R^3 - \frac{8}{9} R^3,
$$

where we have used substitutions $u = R^2 - r^2$ and $v = \cos \theta$. To find the volume of what
remains we subtract the above from \( \frac{4}{3} \pi R^3 \) to yield

\[
\frac{2}{3} \pi R^3 + \frac{8}{3} R^3.
\]
4. (20 pts) Compute the integral
\[ \iint_S \mathbf{F} \cdot \mathbf{n} \, dS, \]  
where \( \mathbf{F} \) is the vector field given by \( \langle x^3, y^3, -z \rangle \). Also \( \mathbf{n} \) is the outward pointing normal vector and \( S \) is the closed surface given by the paraboloid \( z = x^2 + y^2 \) capped off by the plane \( z = 1 \). (This flat top of the region is included as part of \( S \).)

We may use the divergence theorem. Note that \( \text{div} \mathbf{F} = 3x^2 + 3y^2 - 1 = 3r^2 - 1 \) in cylindrical coordinates.

\[ \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_T \text{div} \mathbf{F} \, dV \]
\[ = \int_0^{2\pi} \int_0^1 \int_0^1 (3r^2 - 1)r \, dz \, dr \, d\theta \]
\[ = 2\pi \int_0^1 \int_0^1 (3r^2 - 1)(1 - r^2) \, r \, dr \]
\[ = 0. \]
5. (20 pts) Find the moment of inertia of a circular loop of wire of radius $R$ around an axis tangent to the circle. (I.e. the axis lies in the same plane as the circle and touches the circle as a tangent line.) Assume the density of the wire is constant and express the answer in terms of $R$ and the total mass of the loop of wire.

This distance from the axis is $w = R(1 + \cos \theta)$. Thus

\[ I = \int_C w^2 \delta \, ds \]
\[ = \int_0^{2\pi} R^2 (1 + \cos \theta)^2 \delta R \, d\theta \]
\[ = 3\pi \delta R^3. \]

Since the length of the wire is $2\pi R$, the mass is $m = 2\pi R \delta$. Therefore

\[ I = \frac{3}{2} mR^2. \]